

## **Analysis of Complex Pipe Networks with Multiple Loops and Inlets and Outlets**

The techniques described previously for analysis of pipe flow are satisfactory if the pipe system is simple, consisting of one pipe or a combination of pipes in which the flow directions are all known unambiguously. In more complex systems, pipes might be combined in interconnected loops in ways that make it difficult to determine even the direction of flow in any given pipe. The fundamental relationships that we have derived up to this point – the energy and continuity equations and the relationships between flow and headloss in any given pipe – still apply in such a system, but the sheer number of equations that need to be satisfied to determine the complete flow conditions is daunting. The conditions in such systems are usually solved with specialized computer programs designed specifically for that purpose. However, before such programs were widely available, less sophisticated techniques were developed for analyzing the systems. These techniques are easily programmed into a spreadsheet, so they provide a bridge between the very simple problems that can be solved manually and the massive ones that can be solved only with special software. In this section, we explore these intermediate-scale techniques. These techniques, as well as more sophisticated ones, allow us to answer such questions as:

- \* For the given flow rates, what will be the head loss in each pipe?
- \* Will additional head have to be supplied by pumps in order to obtain the desired flows?
- \* How much will the flow rates change in various parts of the system if a new pipe is installed, connecting two previously unconnected points, or to replace an older, smaller pipe?
- \* How much will the pressure at a consumer's tap drop if the fire hydrant outside his or her home is in use?

### Solving the System of Equations

The methods used historically to solve pipe network problems are “relaxation” techniques, in which a few of the variables are estimated, the others are computed, and the original estimates are revised based on algebraic manipulations of the computed variables. The numerical solution will, of course, depend to some extent on which equation is used to relate flow to headloss.

With respect to pipe network analysis, the traditional approach is known as the Hardy Cross method. This method is applicable if all the pipe sizes (lengths and diameters) are fixed, and either the headlosses between the inlets and outlets are known but the flows are not, or the flows at each inflow and outflow point are known, but the headlosses are not. This latter case is explored next.

The procedure involves making a guess as to the flow rate in each pipe, taking care to make guesses in such a way that the total flow into any junction equals the total flow out of that junction. Then the headloss around each loop is calculated, based on the assumed flows and the selected flow vs. headloss relationship. Next, the system is checked to see if the headloss around each loop is zero. Since the initial flows were guessed, this will probably not be the case. The flow rates are then adjusted in such a way that continuity (mass in equals mass out) is still satisfied at each junction, but the headloss around each loop is closer to zero. This process is repeated until the adjustments are satisfactorily small. The detailed procedure is as follows.

1. Define a set of independent pipe loops in such a way that every pipe in the network is part of at least one loop, and no loop can be represented as a sum or difference of other loops. The easiest way to do this is to choose all of the smallest possible loops in the network.
2. Arbitrarily choose values of  $Q$  in each pipe, such that continuity is satisfied at each pipe junction (sometimes called nodes). Choose a sign convention for each loop; one easy option choice is to consistently define  $Q$  to be positive if the (assumed) direction of flow is clockwise with respect to loop under consideration. This convention means that the same flow in a given pipe might be considered positive when analyzing one loop, and negative when analyzing another.
3. Compute the headloss in each pipe, using the same sign convention for headloss as for flow, so that  $h_f$  in each pipe has the same sign as  $Q$ , when analyzing any given loop.
4. Compute the headloss around each loop. If the headloss around every loop is zero, then all the pipe flow equations are satisfied, and the problem is solved. Presumably, this will not be the case when the initial, arbitrary guesses of  $Q$  are used.
5. Change the flow in each pipe in a given loop by  $\Delta Q$ . By changing the flow rates in all the pipes in a loop by the same amount, we assure that the increase or decrease in the flow into a junction is balanced by the exact same increase or decrease in the flow out, so that we guarantee that the continuity equation is still satisfied. The trick is to make a good guess for what  $\Delta Q$  should be, so that the headloss around the loop approaches zero. To achieve this, we assume that we can choose a value of  $\Delta Q$  that is exactly what is needed to make the headloss zero, and then see how this value of  $\Delta Q$  is expected to be related to other system parameters. The relationship is derived as follows.

The Darcy-Weisbach equation can be written in the form:

$$h_f = f \frac{L}{D} \frac{V^2}{2g} \left( \frac{A^2}{A^2} \right) = f \frac{L}{DA^2} \frac{1}{2g} Q^2 = KQ^2 \quad (1)$$

where  $K$  equals the combination of variables preceding  $Q^2$  in the middle equality. Note that, because  $K$  includes the friction factor, it might change when  $Q$  changes. Nevertheless, for the current calculations, we assume that  $K$  is approximately constant over the range of  $Q$  values of interest.

Consider a pipe in a particular loop in our system. Designate the initial estimate of  $Q$  in that pipe as  $Q_0$ , and the corresponding headloss as  $h_{f0}$ . The new estimate of the flow through the pipe, which we hope will cause the headloss around the loop to be zero, will then be  $Q_1 = Q_0 + \Delta Q$ . We can then estimate the initial (subscript 0) and new (subscript 1) headloss through the pipe to be:

$$h_{f,0} = KQ_0^2 \quad (2)$$

$$h_{f,1} = KQ_1^2 = K(Q_0 + \Delta Q)^2 \quad (3)$$

$$= KQ_0^2 + KQ_0^1\Delta Q + \text{higher order terms in } \Delta Q \quad (4)$$

Assuming that the higher order terms are small compared to the first two, we can drop them from Equation 4. Next, we sum the headlosses in all the pipes around the loop being analyzed. If  $\Delta Q$  was perfectly chosen, then that sum will be zero, so:

$$\sum_{\text{loop}} h_f = 0 = \sum_{\text{loop}} (K_i Q_{i,0}^2 + K_i Q_{i,0} \Delta Q) = \sum_{\text{loop}} K_i Q_{i,0}^2 + n \sum_{\text{loop}} K_i Q_{i,0} \Delta Q \quad (5)$$

where the summation is over all  $i$  pipes in the loop. Since  $\Delta Q$  is the same for every pipe in the loop, it can be taken outside the second summation, yielding:

$$0 = \sum_{\text{loop}} K_i Q_{i,0}^2 + n \Delta Q \sum_{\text{loop}} K_i Q_{i,0} \quad (6)$$

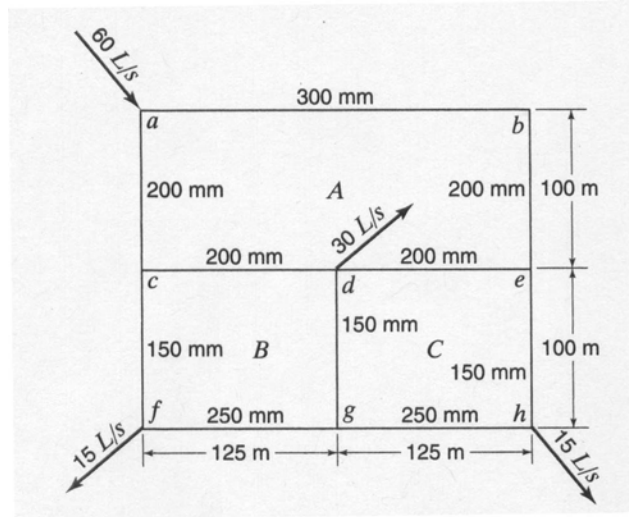
$$\Delta Q = - \frac{\sum_{\text{loop}} KQ_0^2}{2 \sum_{\text{loop}} KQ_0} = - \frac{\sum_{\text{loop}} h_{f,0}}{2 \sum_{\text{loop}} \frac{h_{f,0}}{Q_0}} \quad (7)$$

Equation 7 provides a way to calculate a value of  $\Delta Q$  that will cause the headloss around the loop to be zero, if the higher order terms that are dropped between Equations 4 and 5 are indeed negligible and if  $K$  is the same for flowrates of  $Q$  and  $Q + \Delta Q$ . For the first few iterations, those conditions are not likely to be met, so the computed value of  $\Delta Q$  will not cause the headloss around the loop to be exactly zero, but it will nevertheless cause the headloss to be closer to zero than it was in the previous iteration. The value of  $\Delta Q$  can then be added to the preceding values of  $Q$ , and a new iteration can be carried out. This same process can be used for all the loops in the network, and then repeated until the criterion that  $h_f$  around all the loops be zero is met.

As noted previously, the initial guess of the flow rates is entirely arbitrary, as long as continuity is satisfied at each junction. If one makes good guesses for these flowrates, the problem will converge quickly, and if one makes poor guesses, it will take more iterations before the final solution is found. However, any guesses which meet the mass balance criterion will ultimately lead to the same, correct final result.

Note that the assumptions for flow must be consistent for all the loops in the system. That is, once a flow rate and direction are assumed for a given pipe, those same assumptions must be used for that pipe in every loop that includes it. Finally, note also that, if a pipe is part of two or more different loops, the correction factors for all the loops that contain it are applied to it.

*Example.* Determine the flow rates in all the pipes in the network shown below. Use the DW equation to relate headlosses to flow rates. If the pressure head at point *a* is 40 m, find the pressure head at *d* (which might represent a fire demand, for example).



*Solution.* The analysis can be done in a spreadsheet, as shown below for an initial guess of *Q* values. All values are in units of meters and seconds.

1	2	3	4	5	6	7	8	9
					$\pi D^2/4$	Guess	Q/A	$DV\rho/\mu$
Loop	Pipe	D (m)	L (m)	e/D (—)	A (m <sup>2</sup> )	Q (m <sup>3</sup> /s)	V (m/s)	Re
A	Ab	0.3	250	0.000833	0.071	0.04	0.57	4.19E+04
	Be	0.1	100	0.0025	0.008	0.04	5.09	1.26E+05
	ed	0.2	125	0.00125	0.031	0.025	0.80	4.10E+04
	dc	0.2	125	0.00125	0.031	-0.025	0.80	2.02E+05
	ca	0.2	100	0.00125	0.031	-0.020	0.64	3.19E+05
B	cd	0.2	125	0.00125	0.031	0.025	0.80	2.02E+05
	dg	0.15	100	0.001667	0.018	0.02	1.13	6.99E+04
	gf	0.25	125	0.001	0.049	0.02	0.41	1.71E+04
	fc	0.15	100	0.001667	0.018	0.005	0.28	1.56E+05
C	de	0.2	125	0.00125	0.031	-0.025	0.80	4.10E+04
	eh	0.1	100	0.0025	0.008	0.015	1.91	4.35E+04
	hg	0.25	125	0.001	0.049	0	0.00	5.90E+04
	gd	0.15	100	0.001667	0.018	-0.02	1.13	6.99E+04

1	2	10	11	12	13	14	15	16	17
		From Prob.8-37	f=0 if R=0	DW eqn					
Loop	Pipe	f	ff	$h_f$	$h_f/Q$	$\Sigma h_f$	$\Sigma(h_f/Q)$	$\Delta Q$	new Q
A	ab	0.019	0.019	0.26	6.4				0.0211
	be	0.025	0.025	32.87	821.8				0.0211
	ed	0.021	0.021	0.42	16.7				0.0105
	dc	0.021	0.021	-0.42	16.7				-0.0345
	ca	0.021	0.021	-0.21	10.7	<b>32.91</b>	<b>872.4</b>	<b>-0.0189</b>	-0.0389
B	cd	0.021	0.021	0.42	16.7				0.0345
	dg	0.022	0.022	0.97	48.6				0.0149
	gf	0.020	0.020	0.08	4.2				0.0106
	fc	0.022	0.022	0.06	12.1	<b>1.53</b>	<b>81.6</b>	<b>-0.0094</b>	-0.0044
C	de	0.021	0.021	-0.42	16.7				-0.0105
	eh	0.025	0.025	4.62	308.2				0.0107
	hg	#DIV/0!	0.000	0.00	0.0				-0.0043
	gd	0.022	0.022	-0.97	48.6	<b>3.23</b>	<b>373.5</b>	<b>-0.0043</b>	-0.0149

Columns 1-4 reiterate the given information. Columns 5 and 6 are fixed values that depend only on the given input information. Column 7 is the assumed flow rate through the given pipe for the current iteration. Note that the flow in pipe *cd* is assumed to be from left to right, which is counter-clockwise in Loop A, but clockwise in Loop B, so the flow rate is assigned a negative value in the first case and a positive value in the second one.

Columns 8 and 9 show the computed velocity (without sign) in the pipes and the Reynolds number, for the assumed flow rates, and Column 10 shows the estimate of  $f$  based on the equation given in Munson Problem 8-37; this equation allows one to estimate  $f$  directly from values of  $Re$  and  $\varepsilon/D$ . Because this equation has  $Re$  in the denominator of one term,  $f$  is undefined in pipes with zero flow; in Column 11,  $f$  is assigned a value of zero for such conditions. The value in Column 12 is  $h_f$ , based on the DW equation, with the same sign convention as for flow rate, and Column 13 shows  $h_f/Q$ . Note that, because  $h_f$  and  $Q$  always have the same sign,  $h_f/Q$  is always positive.

Columns 14 and 15 show the summations of  $h_f$  and  $h_f/Q$  around each loop. Column 16 then shows the computed value of  $\Delta Q$  to be applied as a correction to the flow in each pipe. Note that one  $\Delta Q$  is computed for each loop, and that  $\Delta Q$  is applied to every pipe in the loop. Also, pipes that are in more than loop have the corrections for both loops applied to them. Therefore, the formulas for calculating the flow rates to be assumed for the next iteration must be written carefully. For example:

Loop A, pipe *ab*:  $Q_{ab,i+1} = Q_{ab,i} + \Delta Q_{loopA}$   
 Loop A, pipe *dc*:  $Q_{dc,i+1} = Q_{dc,i} + \Delta Q_{loopA} - \Delta Q_{loopB}$   
 Loop B, pipe *cd*:  $Q_{cd,i+1} = Q_{cd,i} - \Delta Q_{loopA} + \Delta Q_{loopB}$   
 Loop B, pipe *dg*:  $Q_{dg,i+1} = Q_{dg,i} + \Delta Q_{loopB} - \Delta Q_{loopC}$

The calculated flow rates and the percentage changes in those rates compared to the previous values for the first and sixth iterations are summarized below. By the sixth iteration, the estimated flow rates are all changing by <5%, which is less than the error associated with the estimate of *f*, so the calculations are as accurate as they will ever get.

<u>Iteration #1</u>		<u>Iteration #6</u>	
<u>new Q</u>	<u>% change</u>	<u>new Q</u>	<u>% change</u>
0.0211	-47.2	0.0098	-0.4
0.0211	-47.2	0.0098	-0.4
0.0105	-75.5	0.0065	-0.6
-0.0345	75.5	-0.0317	0.1
-0.0389	94.3	-0.0502	0.1
0.0345	-37.6	0.0317	-0.4
0.0149	-47.0	0.0082	-1.5
0.0106	-47.0	-0.0035	3.6
-0.0044	-187.9	-0.0185	0.7
-0.0105	17.3	-0.0065	1.6
0.0107	-28.9	0.0033	-3.0
-0.0043	#DIV/0!	-0.0117	0.9
-0.0149	21.6	-0.0082	1.2

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