Economic equivalence

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Economic Equivalence (EE)

- What do we mean by “economic equivalence?”
- Why do we need to establish an economic equivalence?
- How do we measure and compare various cash payments received at different points in time?

Economic Equivalence (EE)

- Economic equivalence exists between cash flows that have the <u>same economic effect</u> and could therefore be traded for one another
- EE refers to the fact that a cash flow-whether a single payment or a series of payments-can be converted to an equivalent cash flow at any point in time
- Even though the amounts and timing of the cash flows may differ, the appropriate interest rate makes them equal in economic sense

Economic Equivalence (EE)

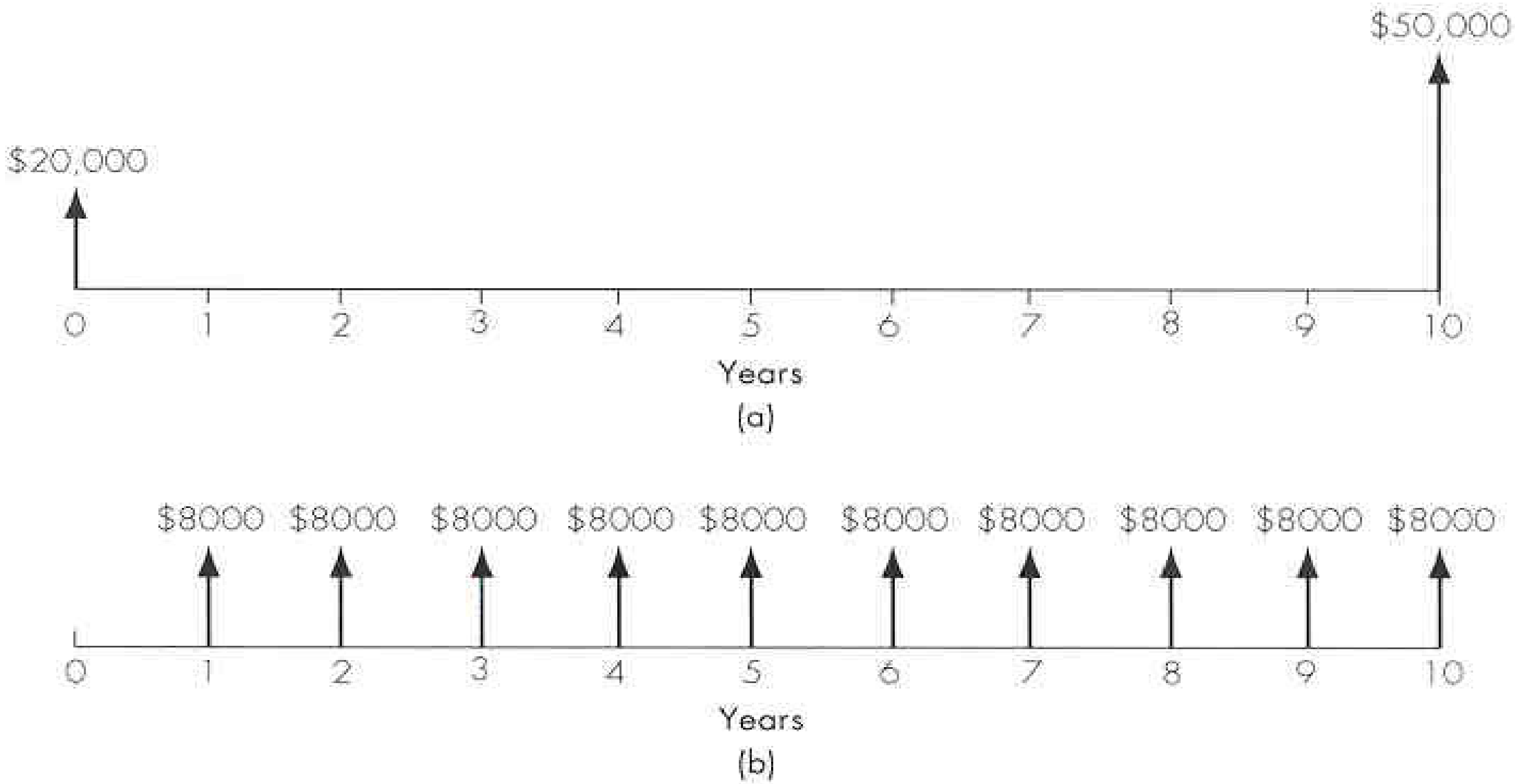


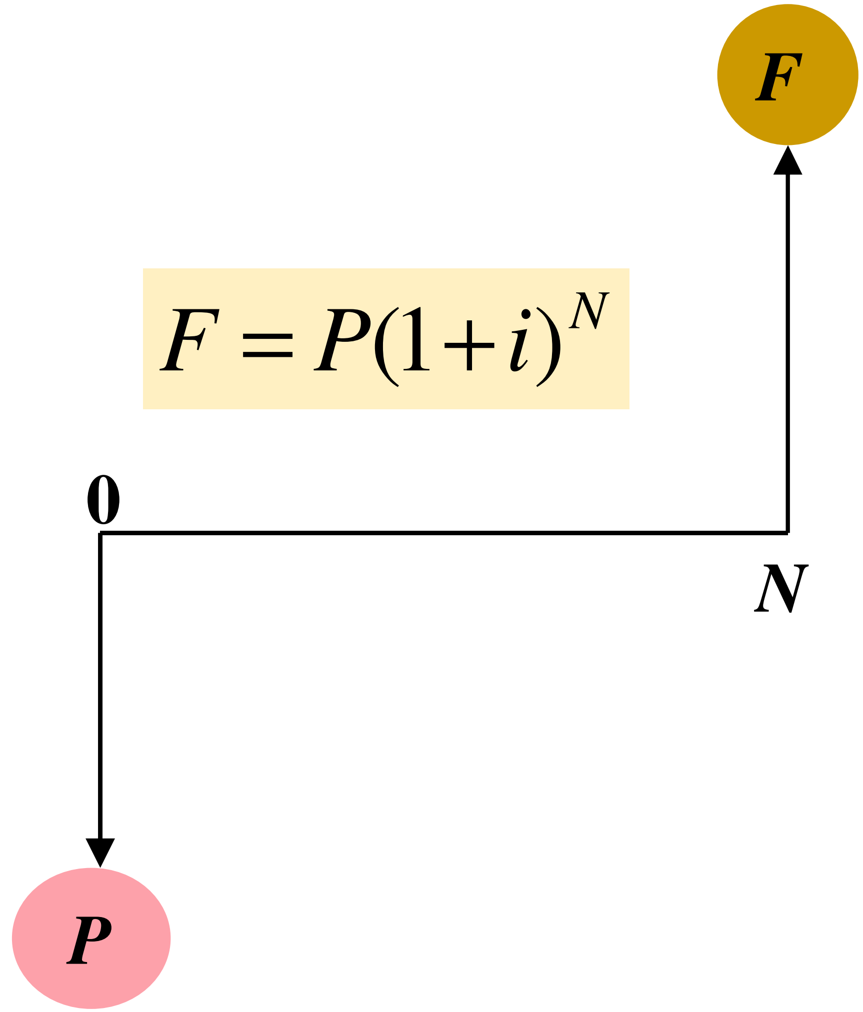
Figure 4.3 Which option would you prefer? (a) Two payments ($20,000 now and $50,000 at the end of 10 years) or (b) ten equal annual receipts in the amount of $8000

Equivalence from Personal Financing Point of View

- If you deposit *P* dollars today for *N* periods at *i*, you will <u>have</u> *F* dollars at the end of period *N*.

$$P \equiv F$$

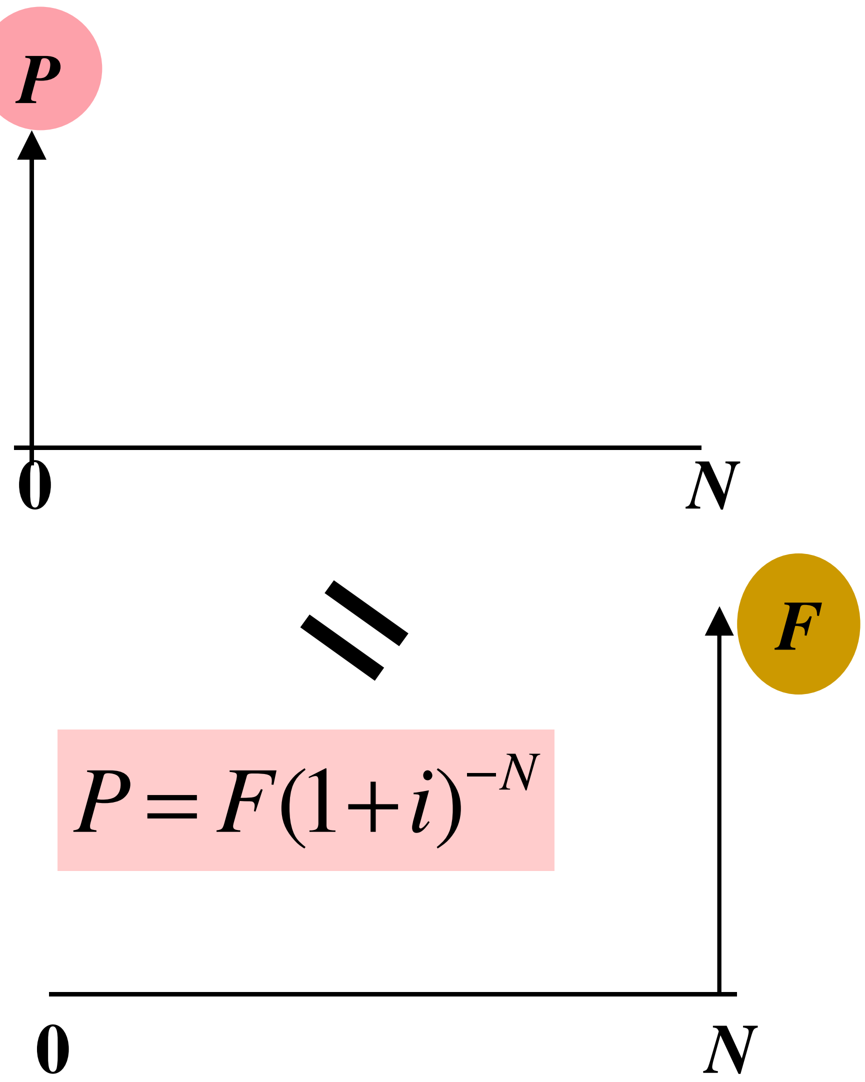
$$F = P(1+i)^N$$



P = present sum/value
F = future sum/value

Alternate Way of Defining Equivalence

- *F* dollars at the end of period *N* is <u>equal</u> to a single sum *P* dollars now, if your earning power is measured in terms of interest rate *i*.



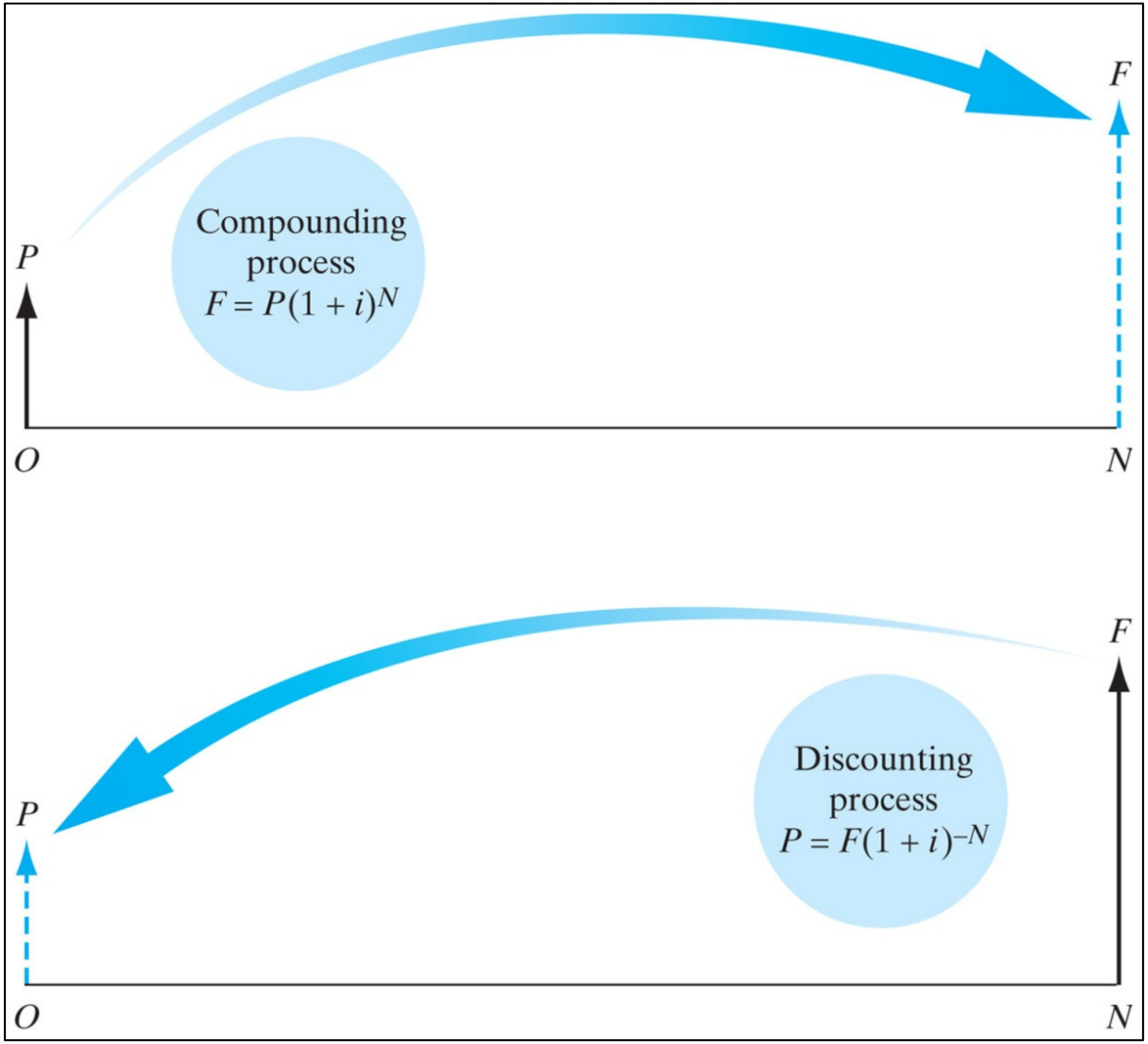
$$P = F(1+i)^{-N}$$

$(1 + i)^{-N}$ = single-payment present-worth factor or discounting factor

Equivalence Relationship Between P and F

- Compounding Process – Finding an equivalent future value of current cash payment

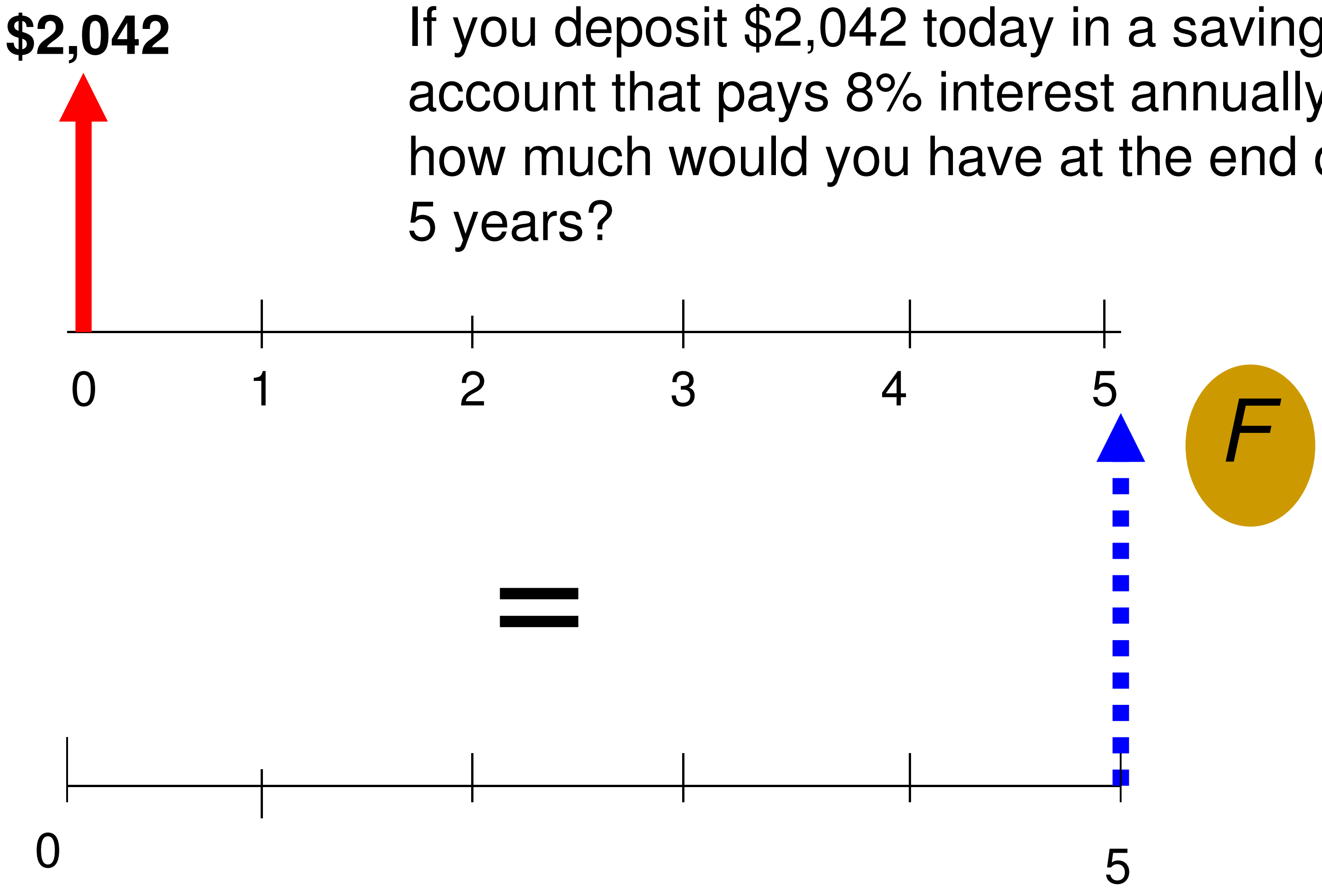
- Discounting Process – Finding an equivalent present value of a future cash payment



Practice Problem (1)

At 8% interest, what is the equivalent worth of $2,042 now 5 years from now?

If you deposit $2,042 today in a savings account that pays 8% interest annually. how much would you have at the end of 5 years?

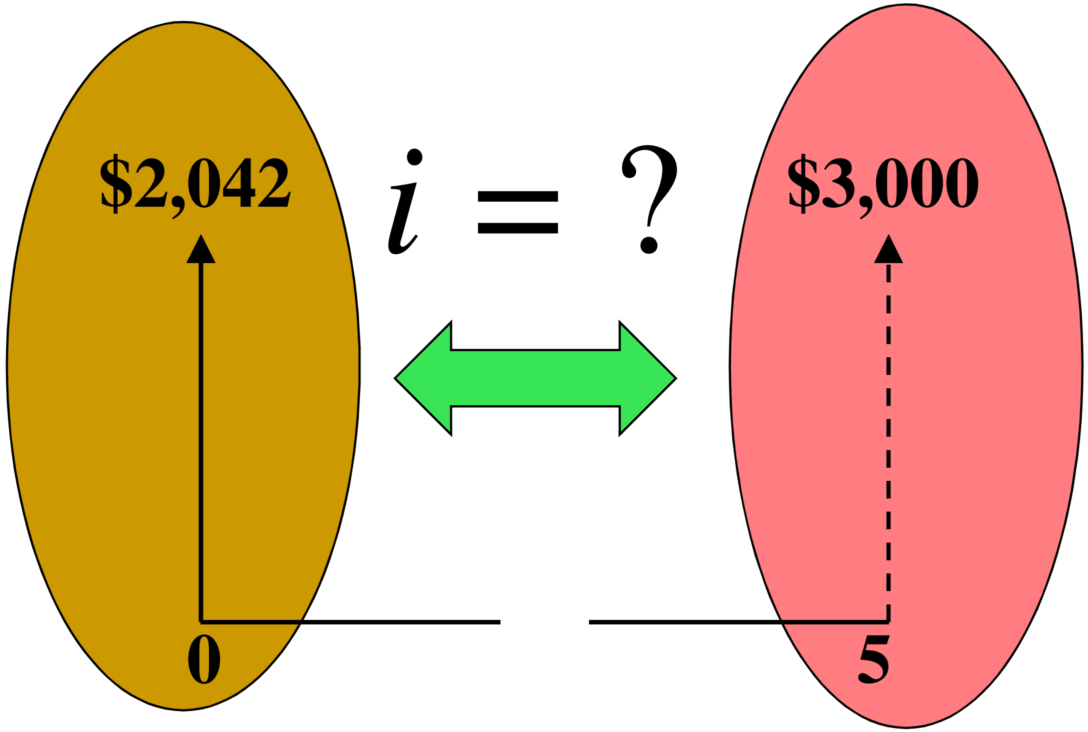


Solution

$$F = \$2,042(1+0.08)^5$$
$$= \$3,000$$

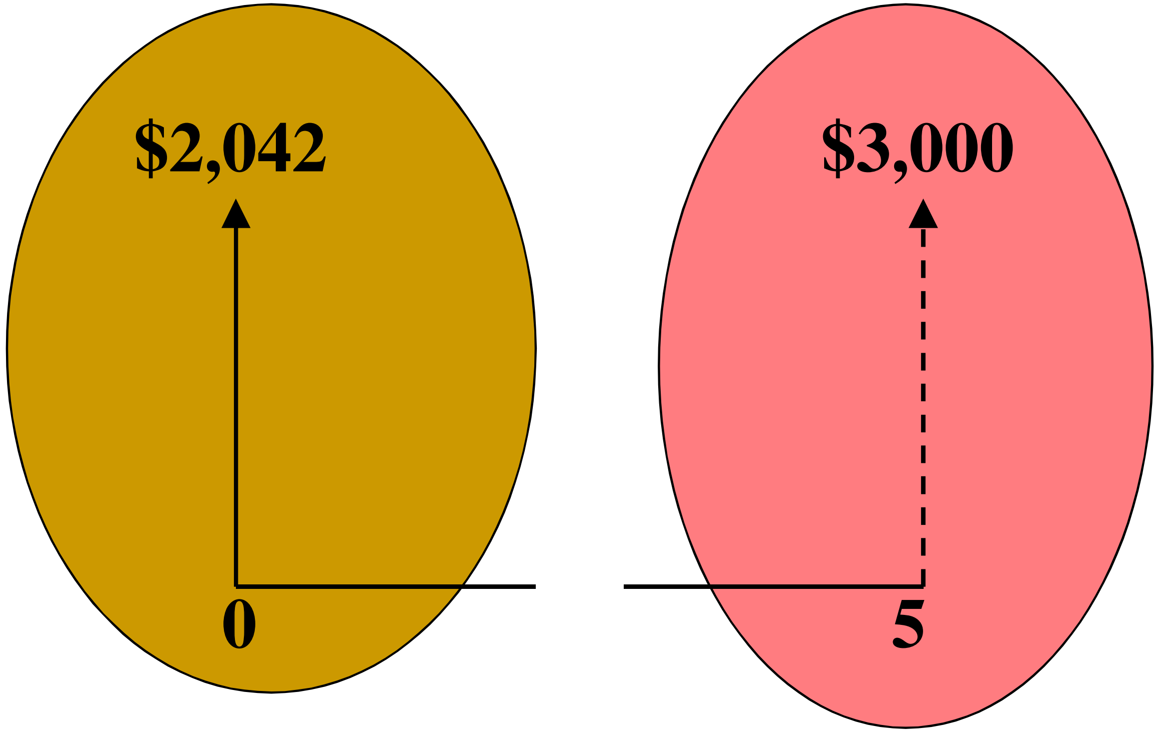
Example (1)

At what interest rate
would these two amounts be equivalent?



Equivalence Between Two Cash Flows

- Step 1: Determine the base period, say, year 5.
- Step 2: Identify the interest rate to use.
- Step 3: Calculate equivalence value.



$$i = 6\%,\ F = \$2{,}042(1 + 0.06)^5 = \$2{,}733$$

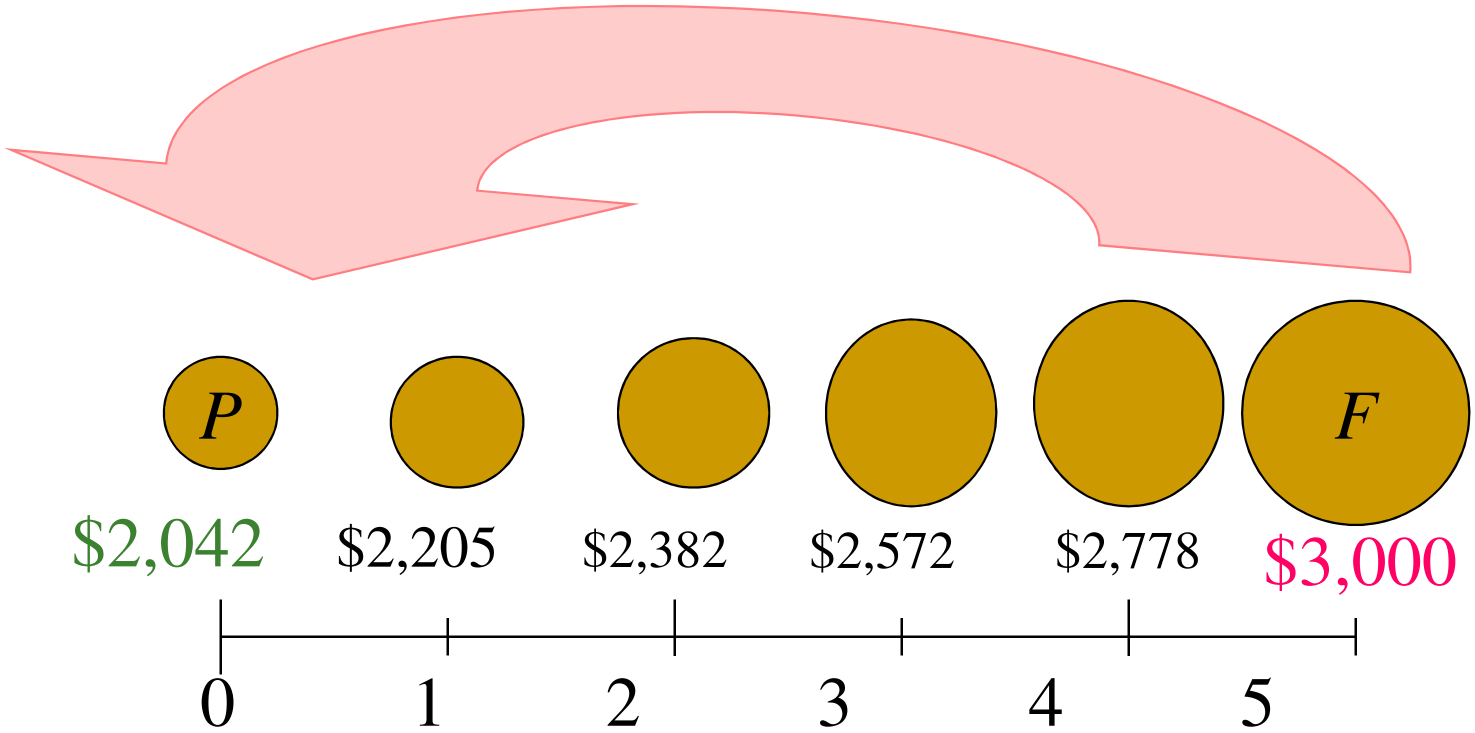
$$i = 8\%,\ F = \$2{,}042(1 + 0.08)^5 = \$3{,}000$$

$$i = 10\%,\ F = \$2{,}042(1 + 0.10)^5 = \$3{,}289$$

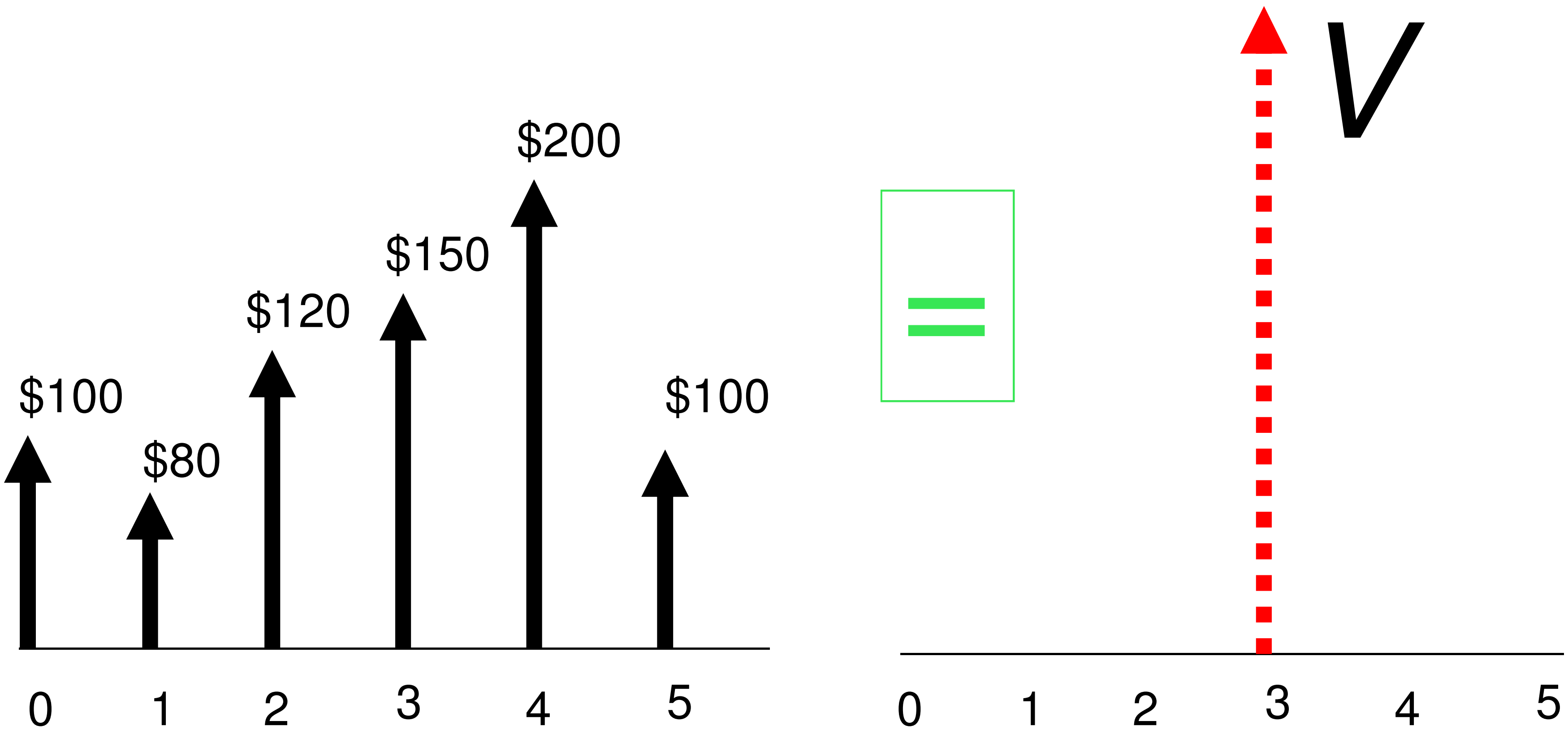
Example - Equivalence

Various dollar amounts that will be economically equivalent to \$3,000 in 5 years, given an interest rate of 8%.

$$P = \frac{\$3{,}000}{(1+0.08)^5} = \$2{,}042$$

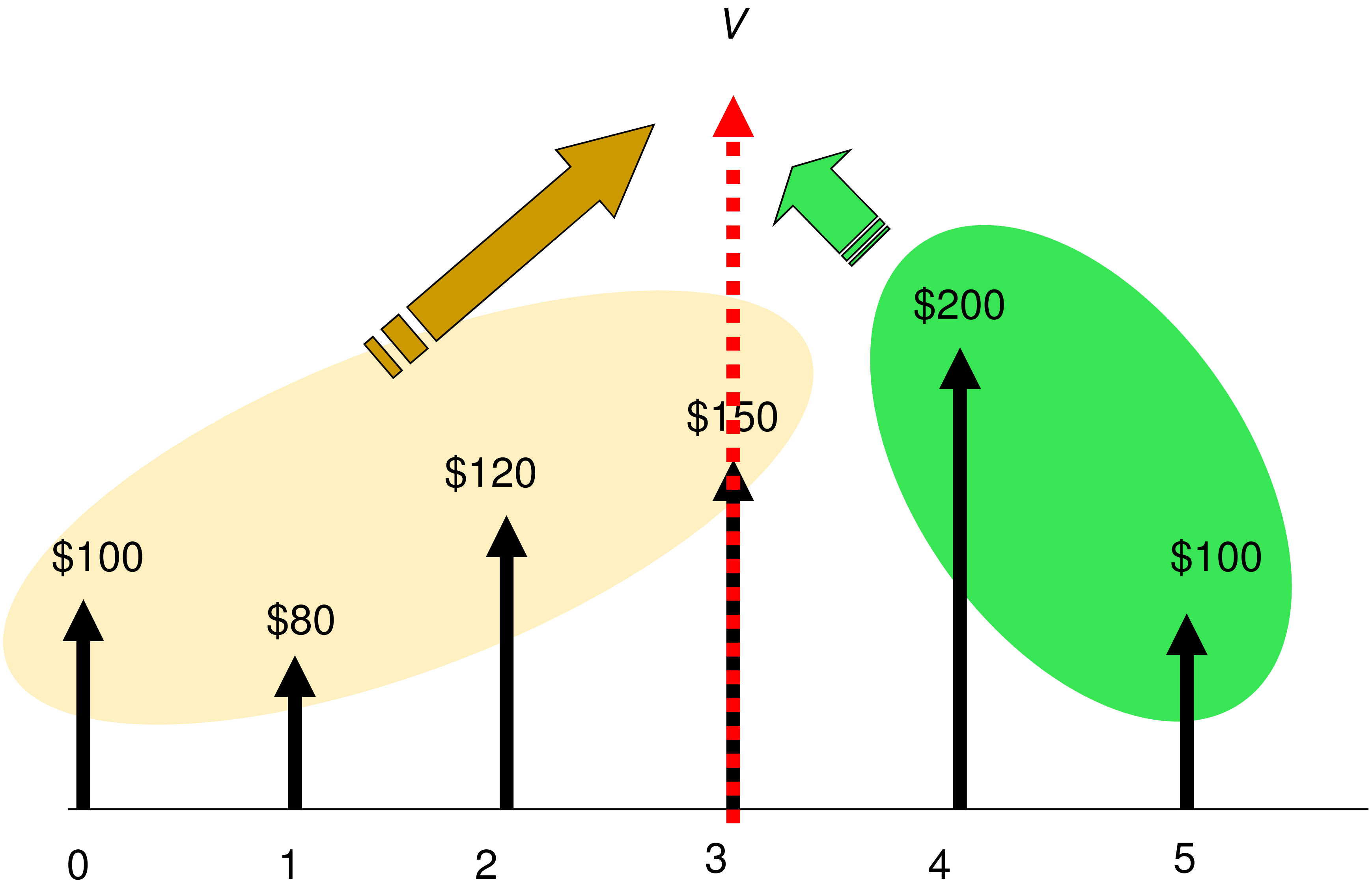


Example (2)

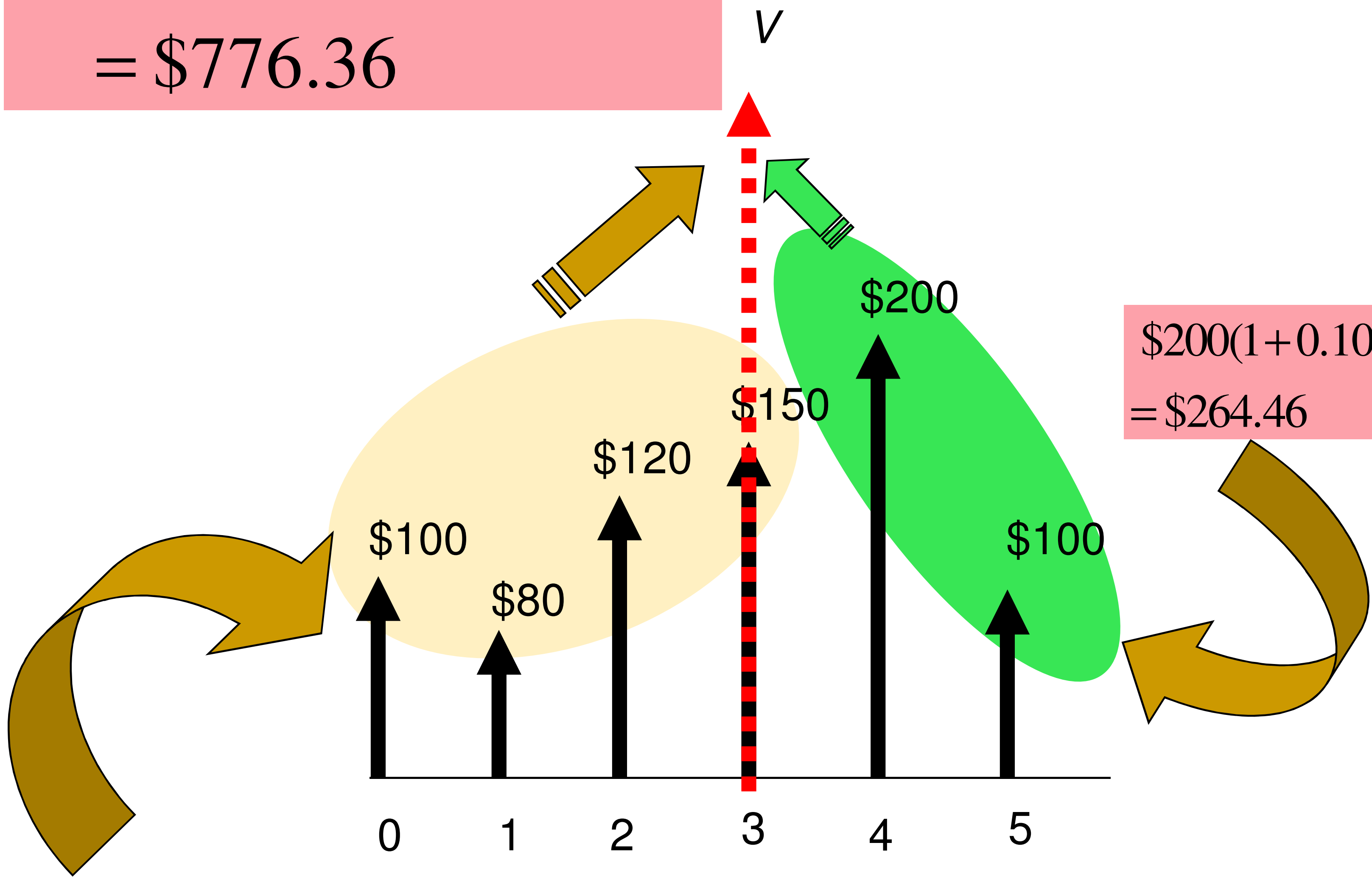


Compute the equivalent lump-sum amount at $n = 3$ at 10% annual interest.

Approach



$$V_3 = \$511.90 + \$264.46$$
$$= \$776.36$$

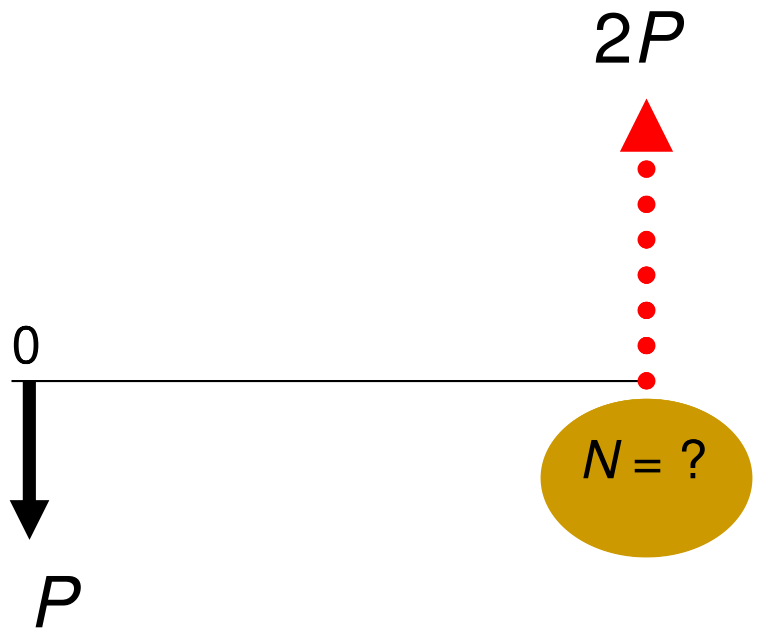


$$\$200(1+0.10)^{-1} + \$100(1+0.10)^{-2}$$
$$= \$264.46$$

$$100(1+0.10)^3 + \$80(1+0.10)^2 + \$120(1+0.10) + \$150$$
$$= \$511.90$$

Practice Problem (2)

- How many years would it take an investment to double at 10% annual interest?



- Solution:

$$F = 2P = P(1+0.10)^N$$

$$2 = 1.1^N$$

$$\log 2 = N \log 1.1$$

$$N = \frac{\log 2}{\log 1.1}$$

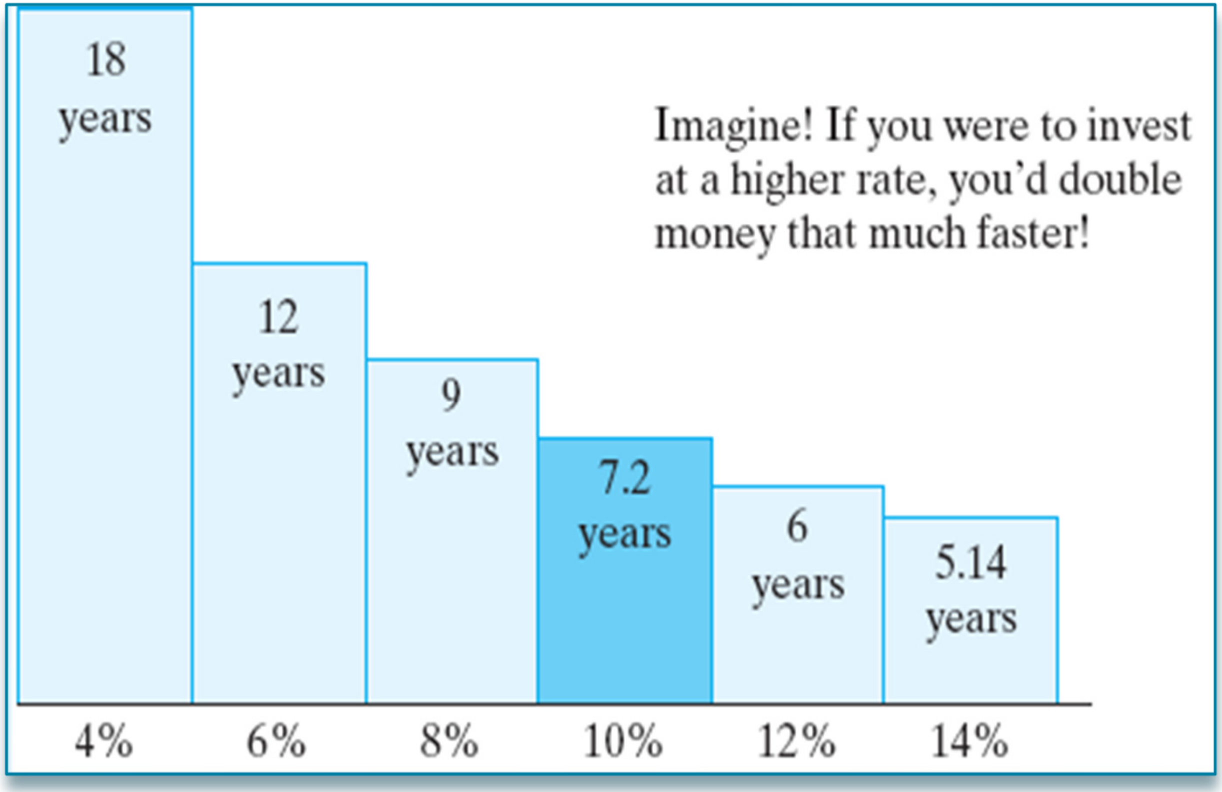
$$= 7.27 \text{ years}$$

Hints: “Rule of 72”

- Approximating how long it will take for a sum of money to double

$$N \cong \frac{72}{\text{interest rate (\%)}} = \frac{72}{20} = 3.6 \text{ years}$$

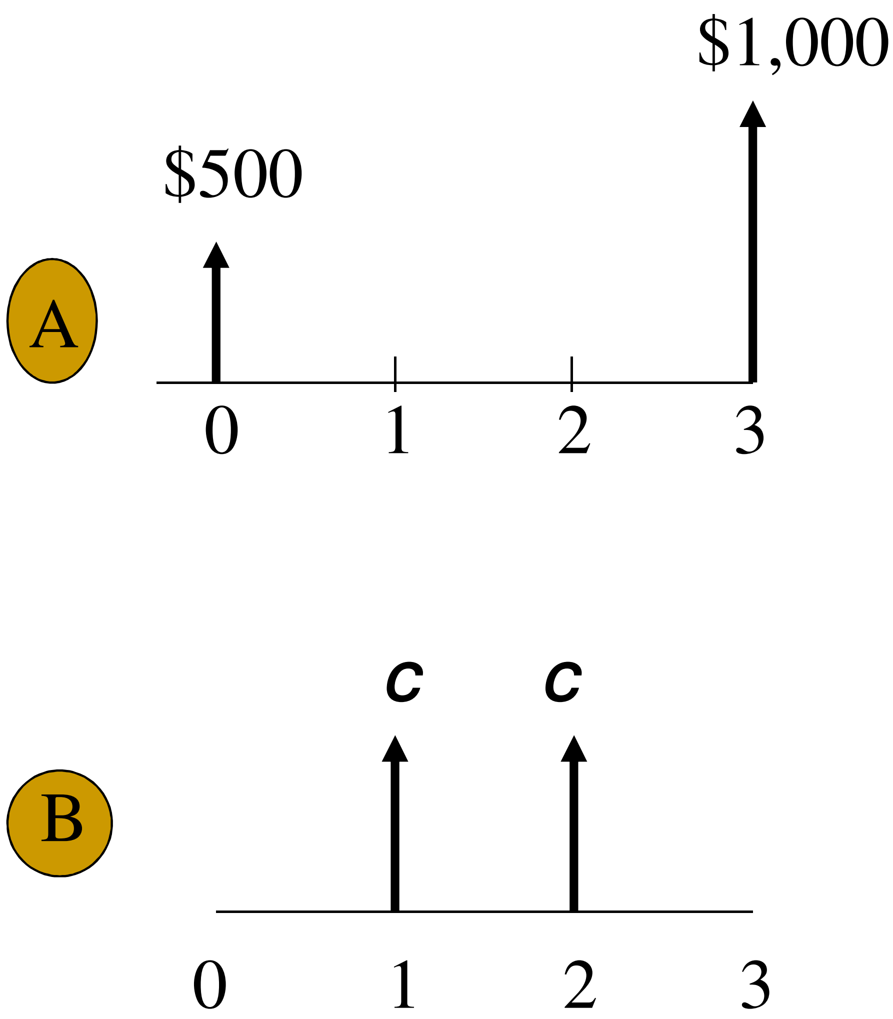
Number of years required to double an initial investment at various interest rates:



Practice Problem (3)

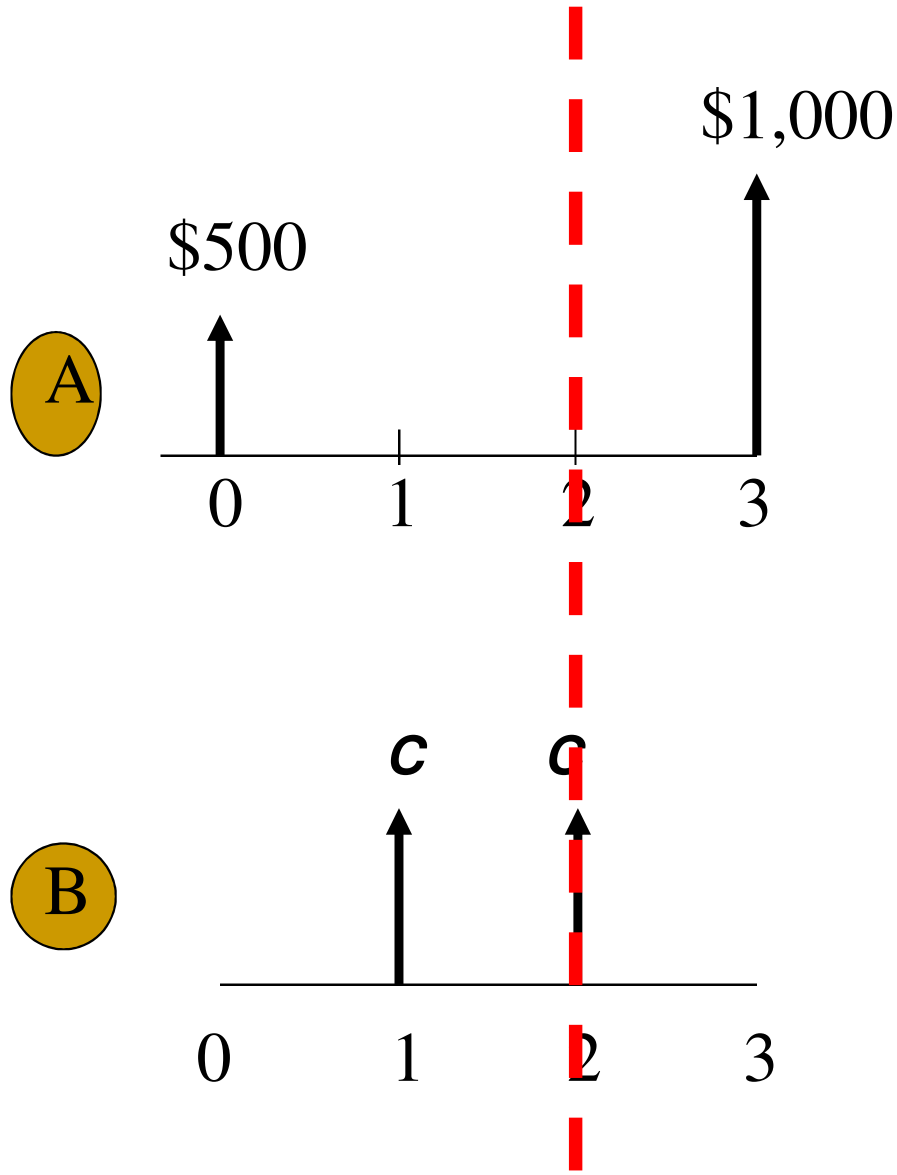
Given: $i = 10\%$,

Find: C that makes the two cash flow streams to be indifferent



Approach

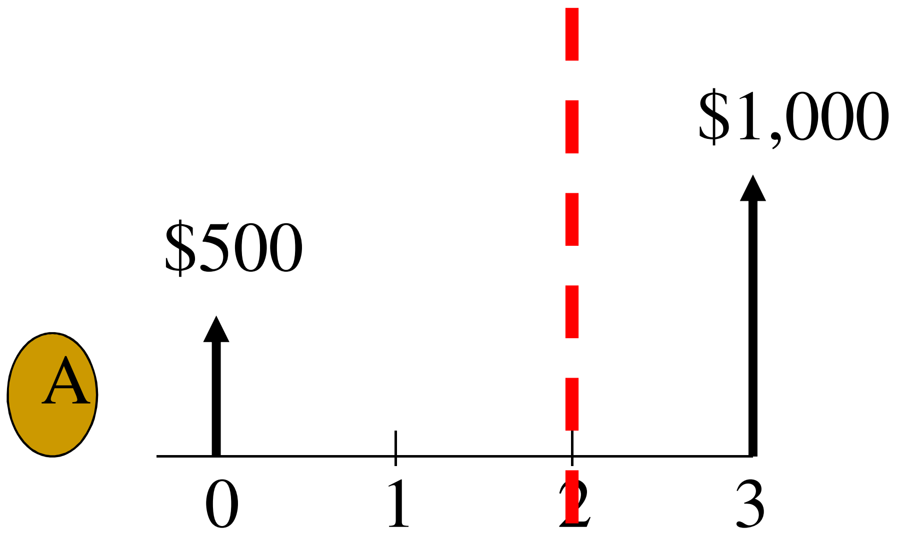
- Step 1: Select the base period to use, say n = 2.
- Step 2: Find the equivalent lump sum value at n = 2 for both A and B.
- Step 3: Equate both equivalent values and solve for unknown C.



Solution

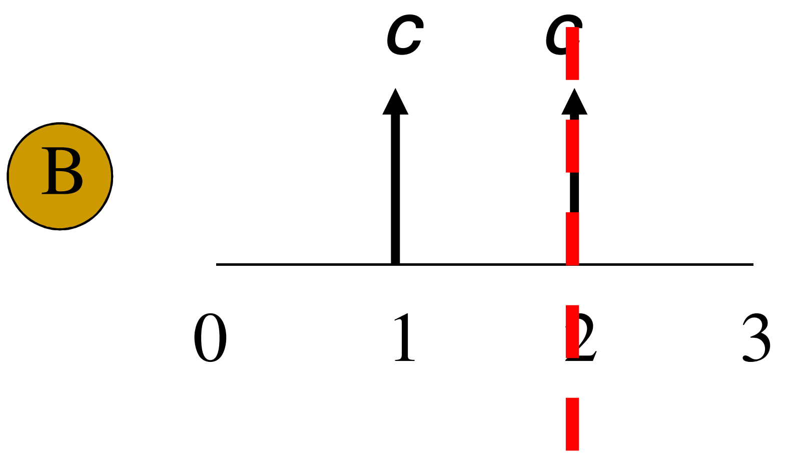
- For A:

$$V_2 = \$500(1+0.10)^2 + \$1,000(1+0.10)^{-1}$$
$$= \$1,514.09$$



- For B:

$$V_2 = C(1+0.10) + C$$
$$= 2.1C$$

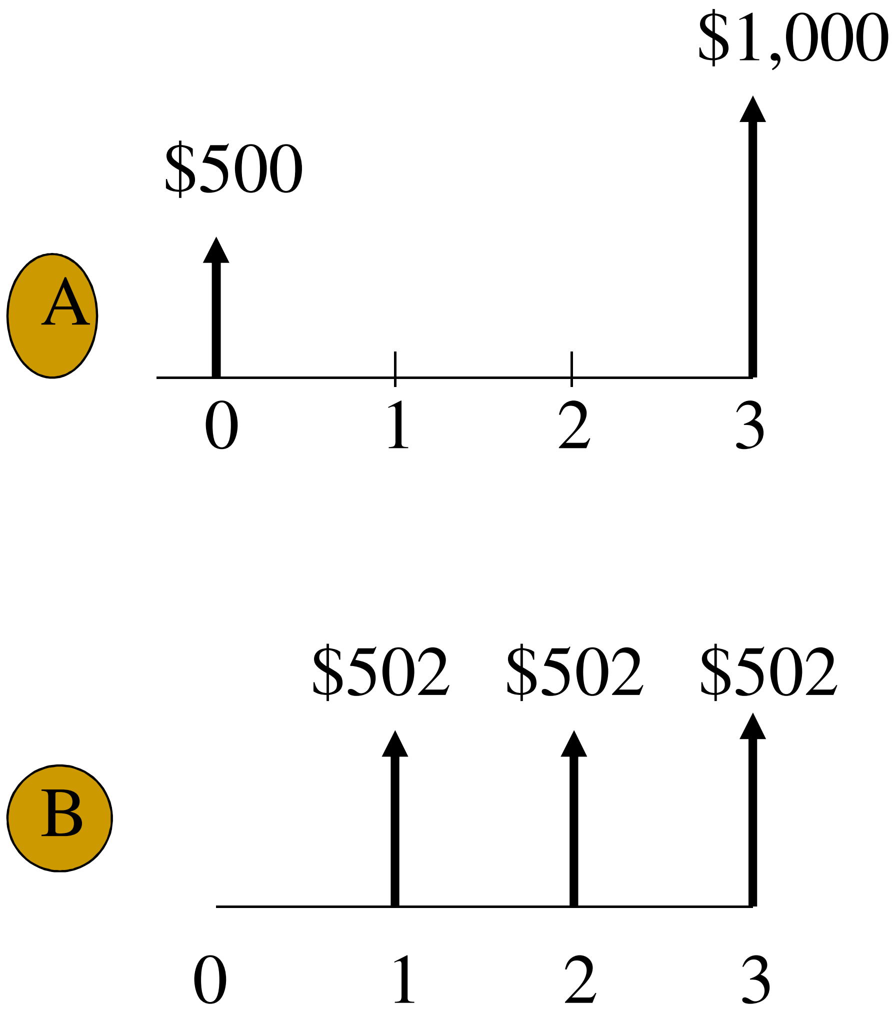


- To Find C:

$$2.1C = \$1,514.09$$
$$C = \$721$$

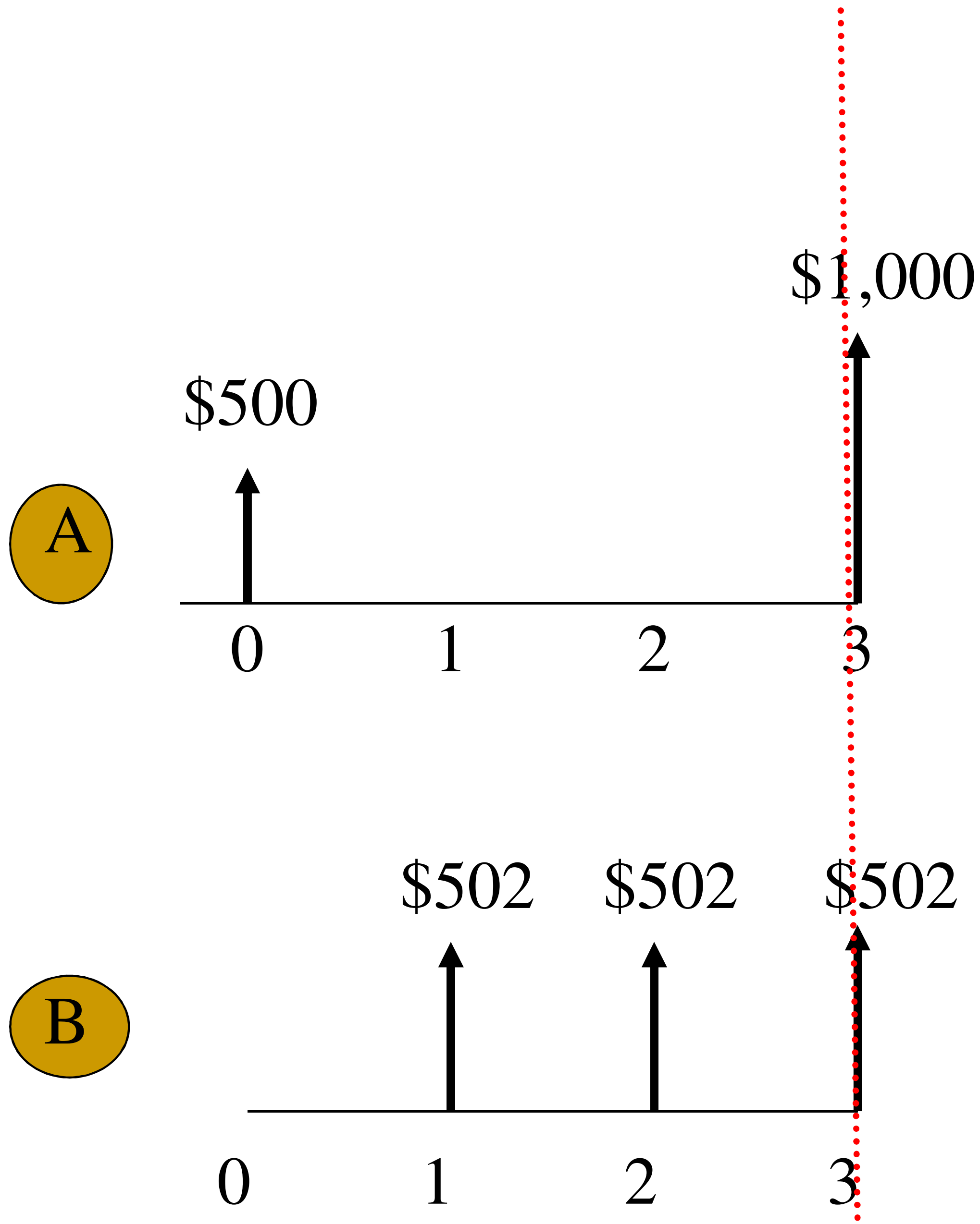
Practice Problem (4)

At what interest rate would you be indifferent between the two cash flows?



Approach

- Step 1: Select the base period to compute the equivalent value (say, $n = 3$)
- Step 2: Find the net worth of each at $n = 3$.



Establish Equivalence at $n = 3$

$$\text{Option A}: F_3 = \$500(1+i)^3 + \$1{,}000$$

$$\text{Option B}: F_3 = \$502(1+i)^2 + \$502(1+i) + \$502$$

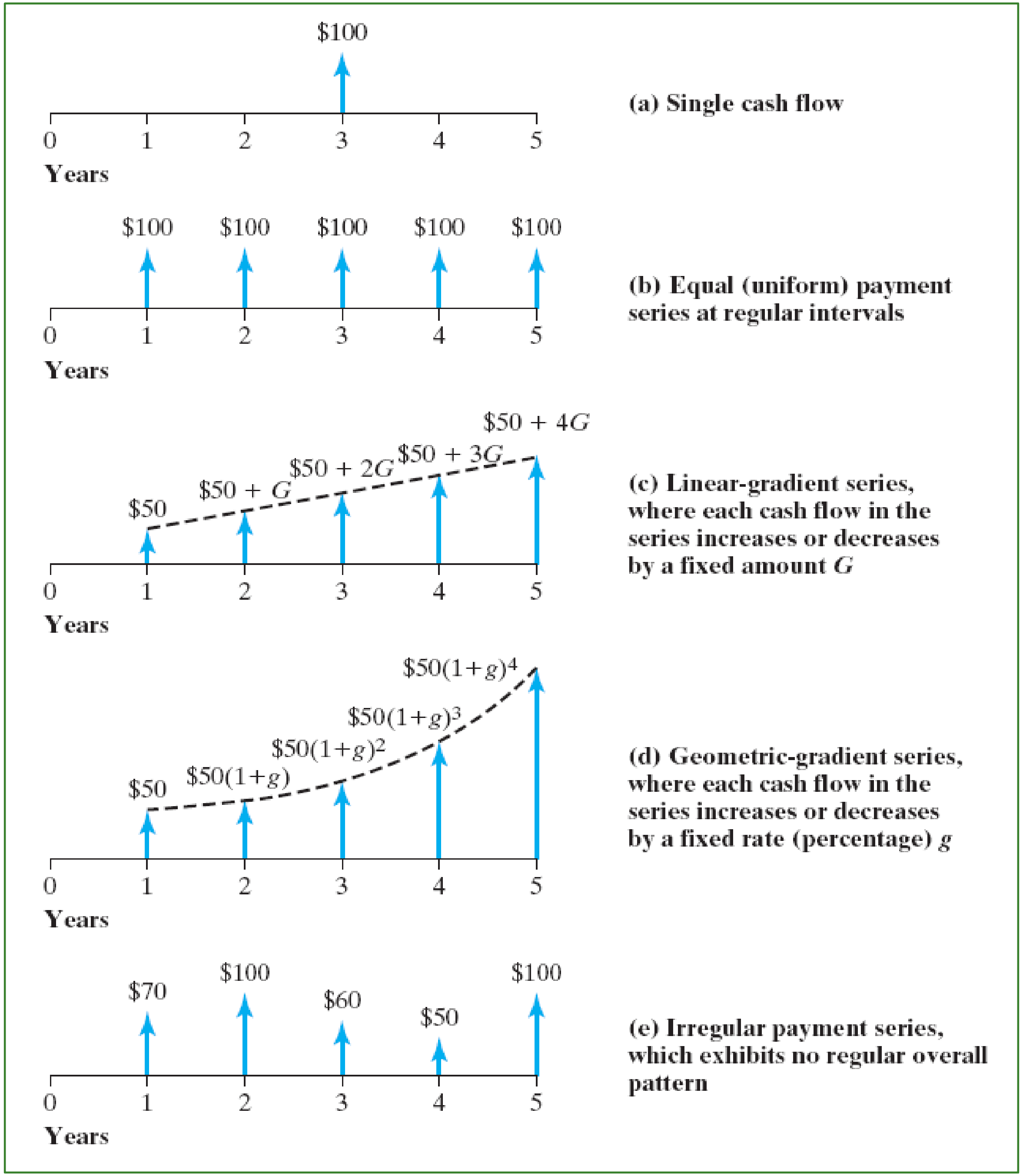
- Find the solution by trial and error, say $i = 8\%$

$$\begin{aligned}\text{Option A}: F_3 &= \$500(1.08)^3 + \$1{,}000 \\ &= \$1{,}630\end{aligned}$$

$$\begin{aligned}\text{Option B}: F_3 &= \$502(1.08)^2 + \$502(1.08) + \$502 \\ &= \$1{,}630\end{aligned}$$

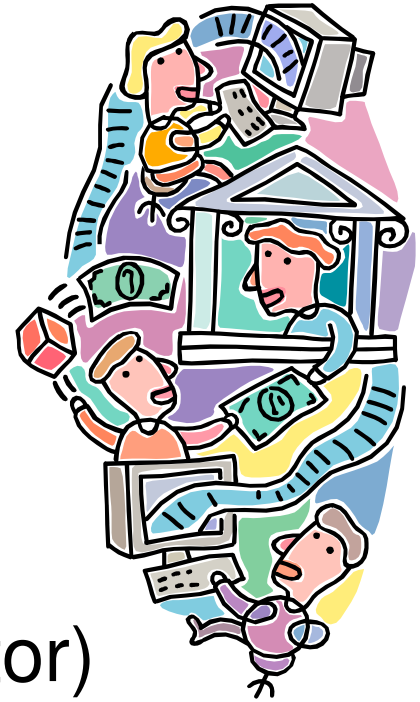
5 Types of Common Cash Flows

- 1. Single cash flow
- 2. Equal (uniform) payment series at regular intervals
- 3. Linear gradient series
- 4. Geometric gradient series
- 5. Irregular (mixed) payment series



Cash Flow & Interest Formulas

- Single Cash Flow
- Multiple (Uneven) Payments
- Equal Payment (Uniform) Series
  - Compound Amount Factor
  - Finding an Annuity Value
  - Sinking Fund
  - Capital Recovery Factor (Annuity Factor)
  - Present Worth of Annuity Series
- Linear Gradient Series
- Geometric Gradient Series



Single Cash Flow Formula
(Find *F*, Given *i*, *N*, and *P*)

- <u>Single payment compound amount factor</u> (growth factor)

Formula

$$\boxed{F = P(1+i)^N}$$

$$\boxed{F = P(F/P, i, N)}$$

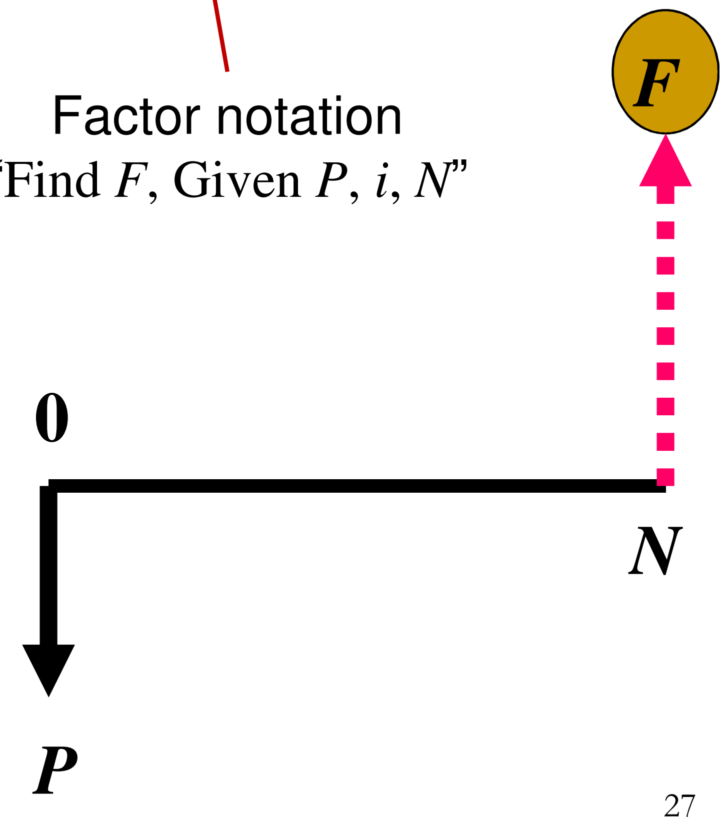
Factor notation
"Find *F*, Given *P*, *i*, *N*"

- Given:

$$i = 10\%$$
$$N = 8 \text{ years}$$
$$P = \$2{,}000$$

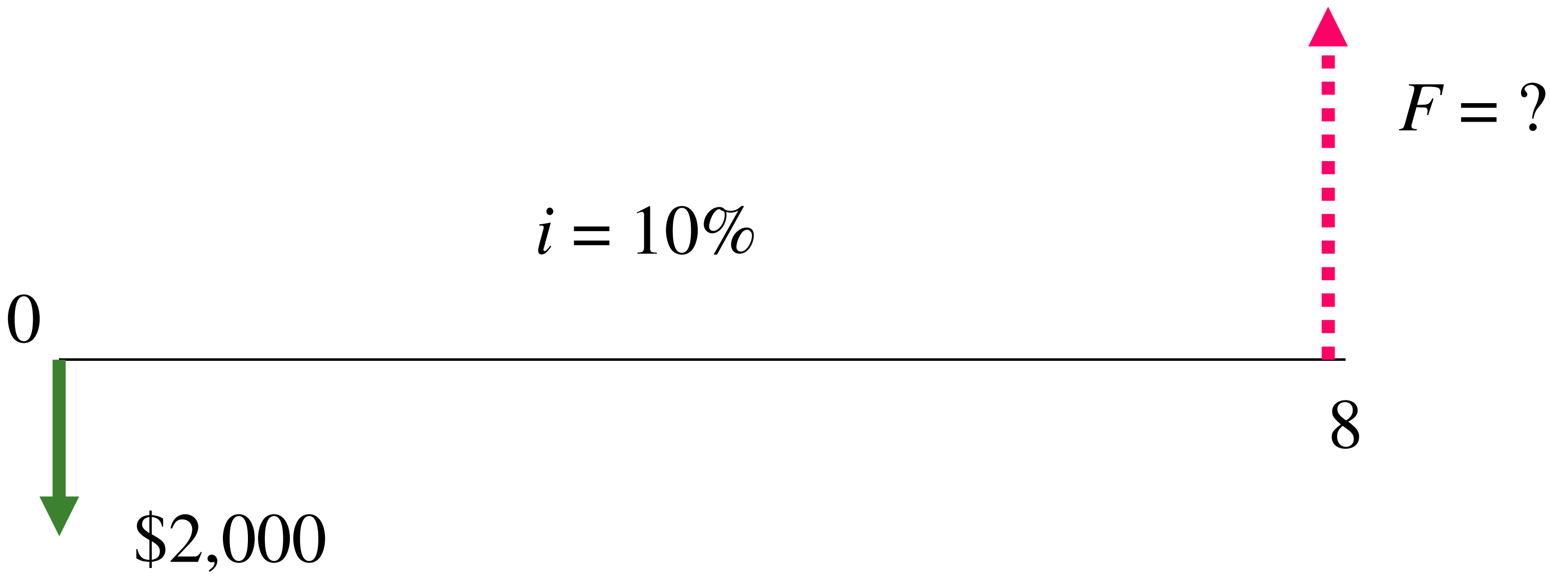
- Find: *F*

$$F = \$2{,}000(1+0.10)^8$$
$$= \$2{,}000(F/P, 10\%, 8)$$
$$= \$4{,}287.18$$



Practice Problem (5)

- If you had $2,000 now and invested it at 10%, how much would it be worth in 8 years?



Solution

Given:

$$P = \$2,000$$

$$i = 10\%$$

$$N = 8 \text{ years}$$

Find: F

$$\begin{aligned} F &= \$2,000(1+0.10)^8 \\ &= \$2,000(F/P, 10\%, 8) \\ &= \$4,287.18 \end{aligned}$$

EXCEL command:

$$\begin{aligned} &= \mathrm{FV}(10\%, 8, 0, 2000, 0) \\ &= \$4,287.20 \end{aligned}$$

Single Cash Flow Formula
(Find *P*, Given *i*, *N*, and *F*)

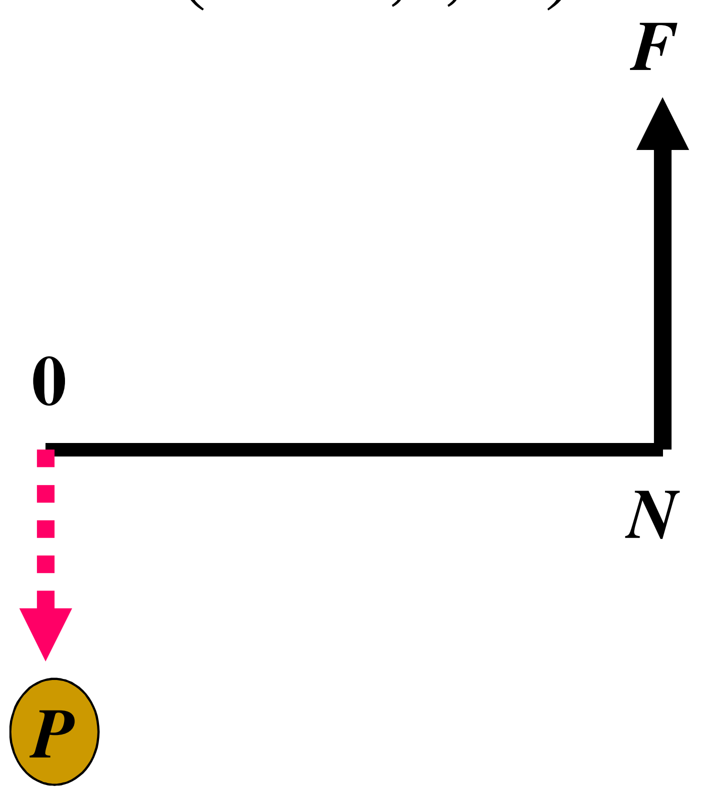
- Single payment present worth factor (discount factor)
- Given:

$$i = 12\%$$
$$N = 5 \text{ years}$$
$$F = \$1{,}000$$

- Find:

$$P = \$1{,}000(1+0.12)^{-5}$$
$$= \$1{,}000(P/F, 12\%, 5)$$
$$= \$567.40$$

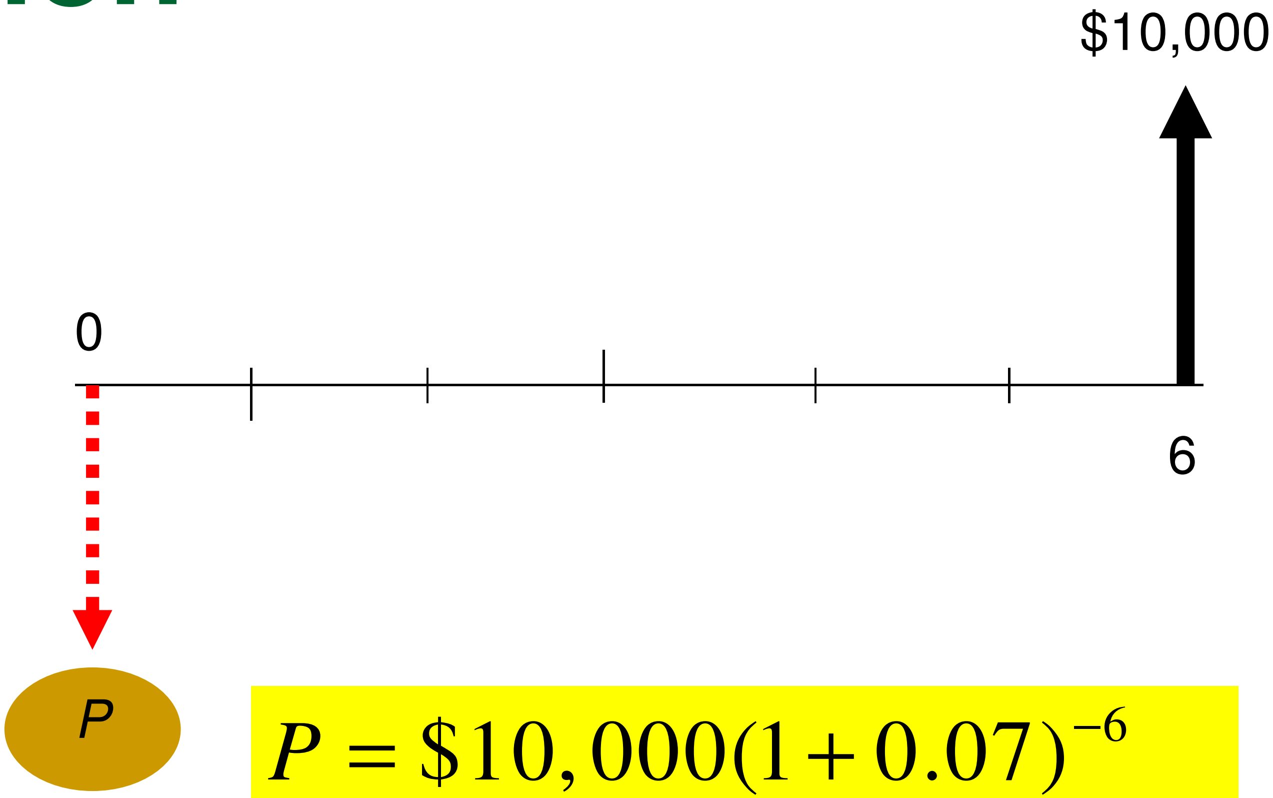
$$P = F(1+i)^{-N}$$
$$P = F(P/F, i, N)$$



Practice Problem (6)

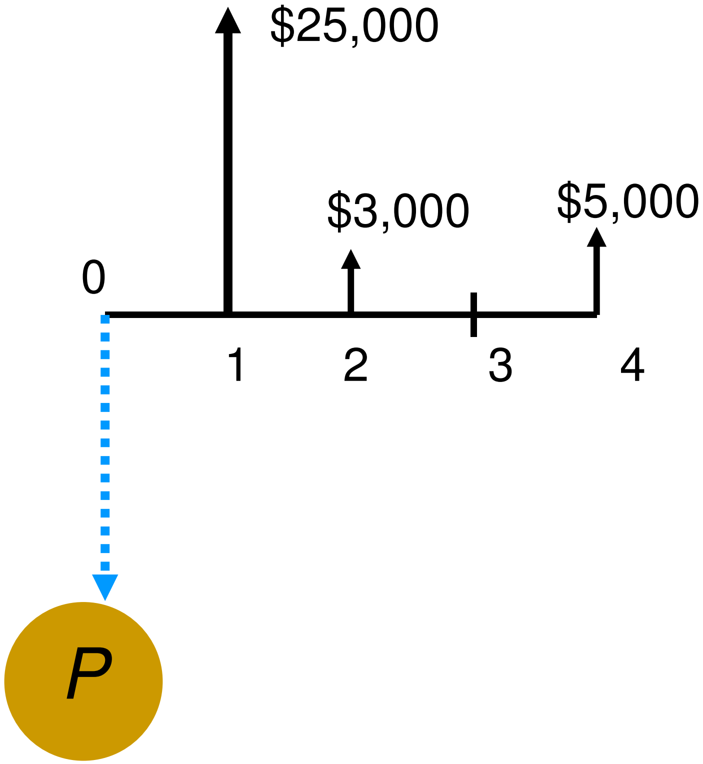
- You want to set aside *a lump sum* amount today in a savings account that earns 7% annual interest to meet a future expense in the amount of $10,000 to be incurred in 6 years. How much do you need to deposit today?

Solution



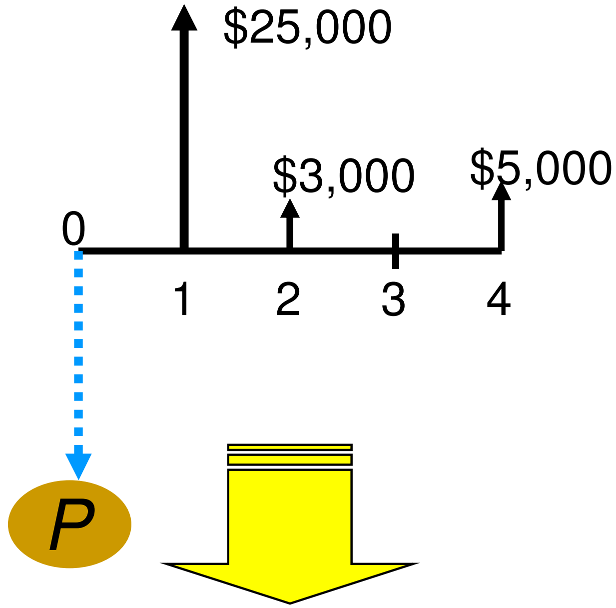
$$P = \$10,000(1+0.07)^{-6}$$
$$= \$10,000(P/F, 7\%, 6)$$
$$= \$6,663$$

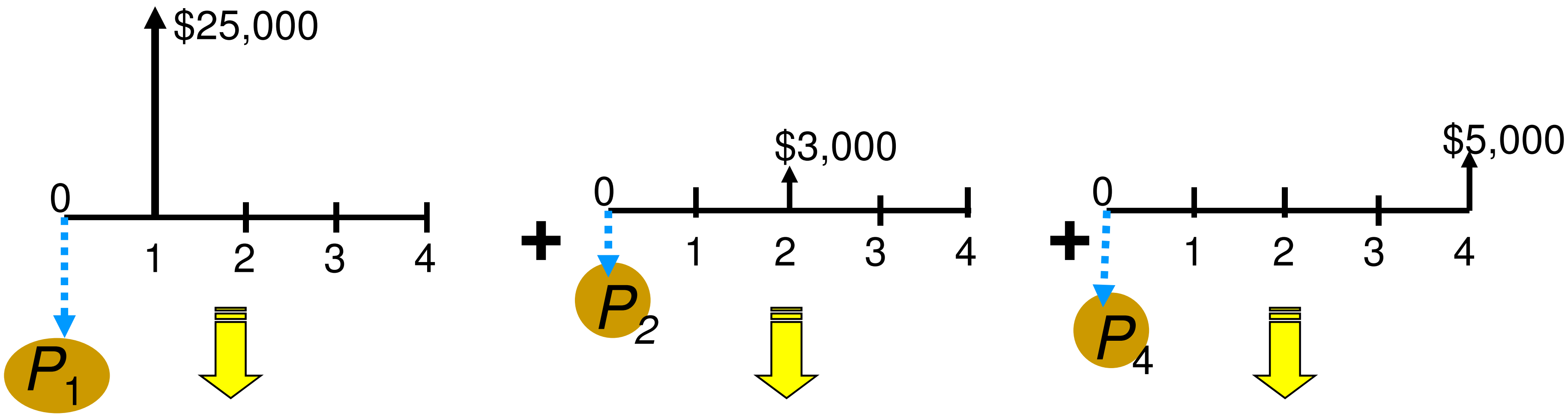
Multiple (Uneven) Payments



- How much do you need to deposit today (*P*) to withdraw \$25,000 at $n = 1$, \$3,000 at $n = 2$, and \$5,000 at $n = 4$, if your account earns 10% annual interest?

Uneven Payment Series





$$P_1 = \$25,000(P/F, 10\%, 1) = \$22,727$$

$$P_2 = \$3,000(P/F, 10\%, 2) = \$2,479$$

$$P_4 = \$5,000(P/F, 10\%, 4) = \$3,415$$

$$P = P_1 + P_2 + P_3 = \$28,622$$

Uneven Payment Series

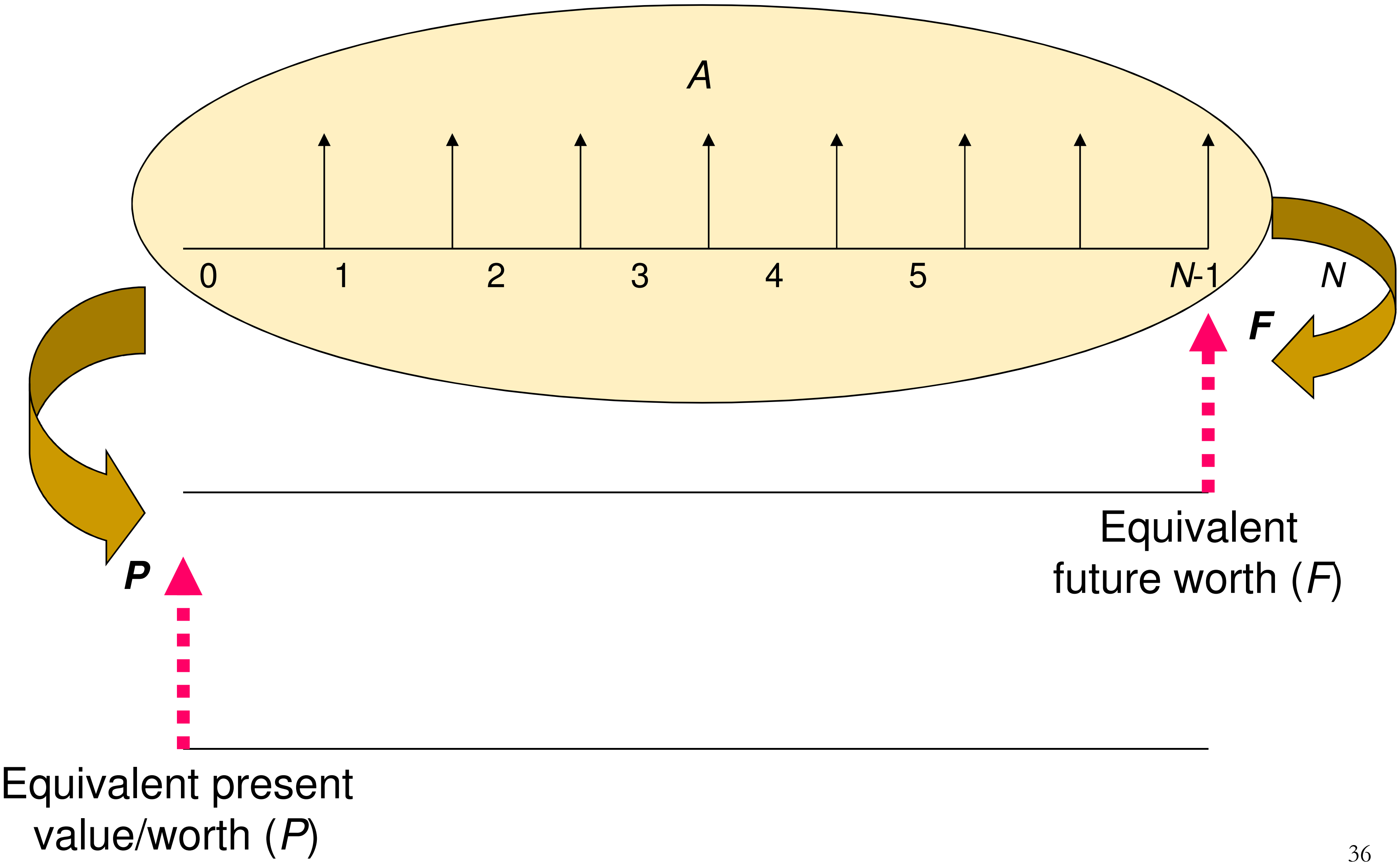
Check the answer again:

	0	1	2	3	4
Beginning Balance	0	28,622	6,484.20	4,132.62	4,545.88
Interest Earned (10%)	0	2,862	648.42	413.26	454.59
Payment	+28,622	-25,000	-3,000	0	-5,000
Ending Balance	$28,622	6,484.20	4,132.62	4,545.88	0.47

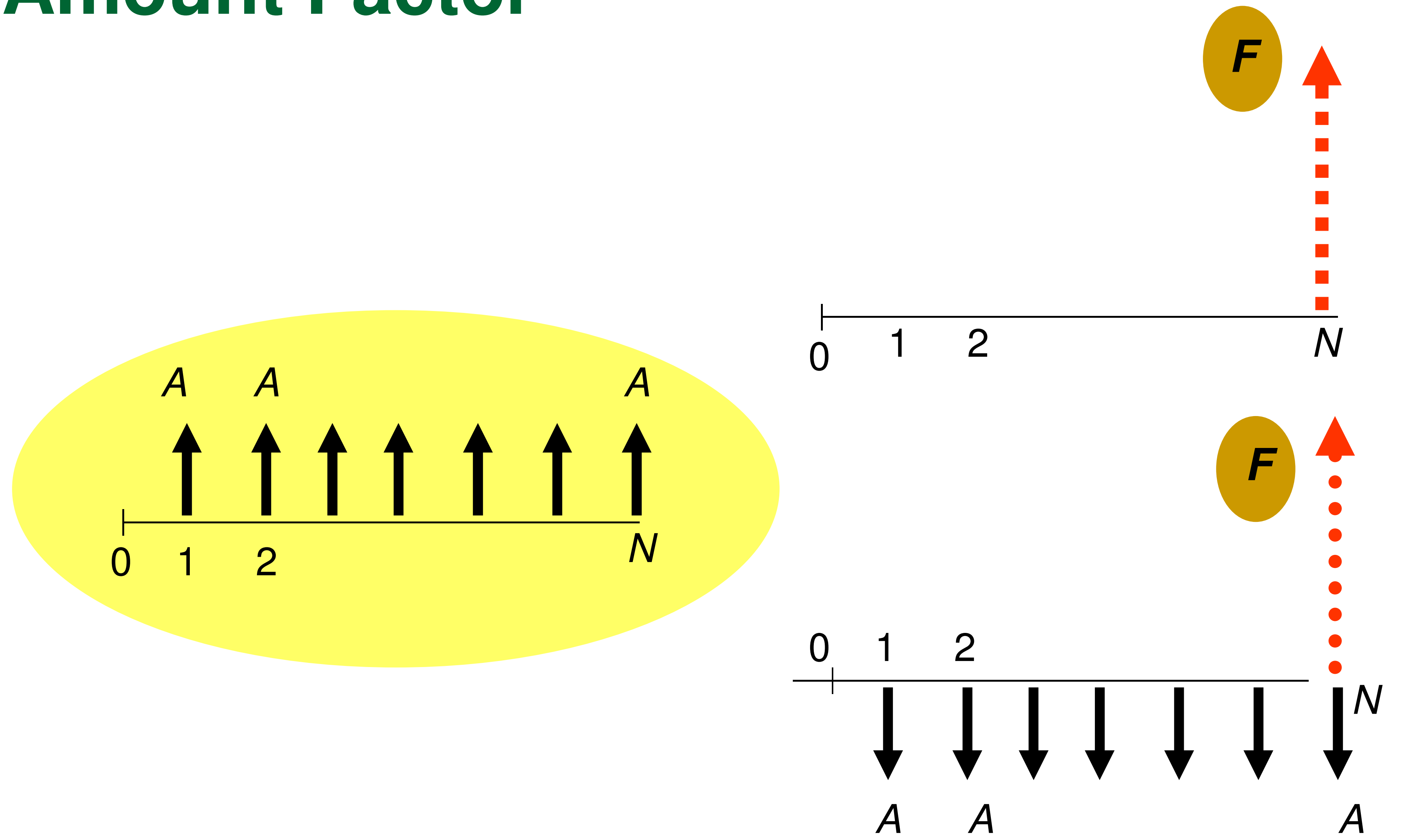
Rounding error
It should be “0.”

Equal Payment (Uniform) Series:

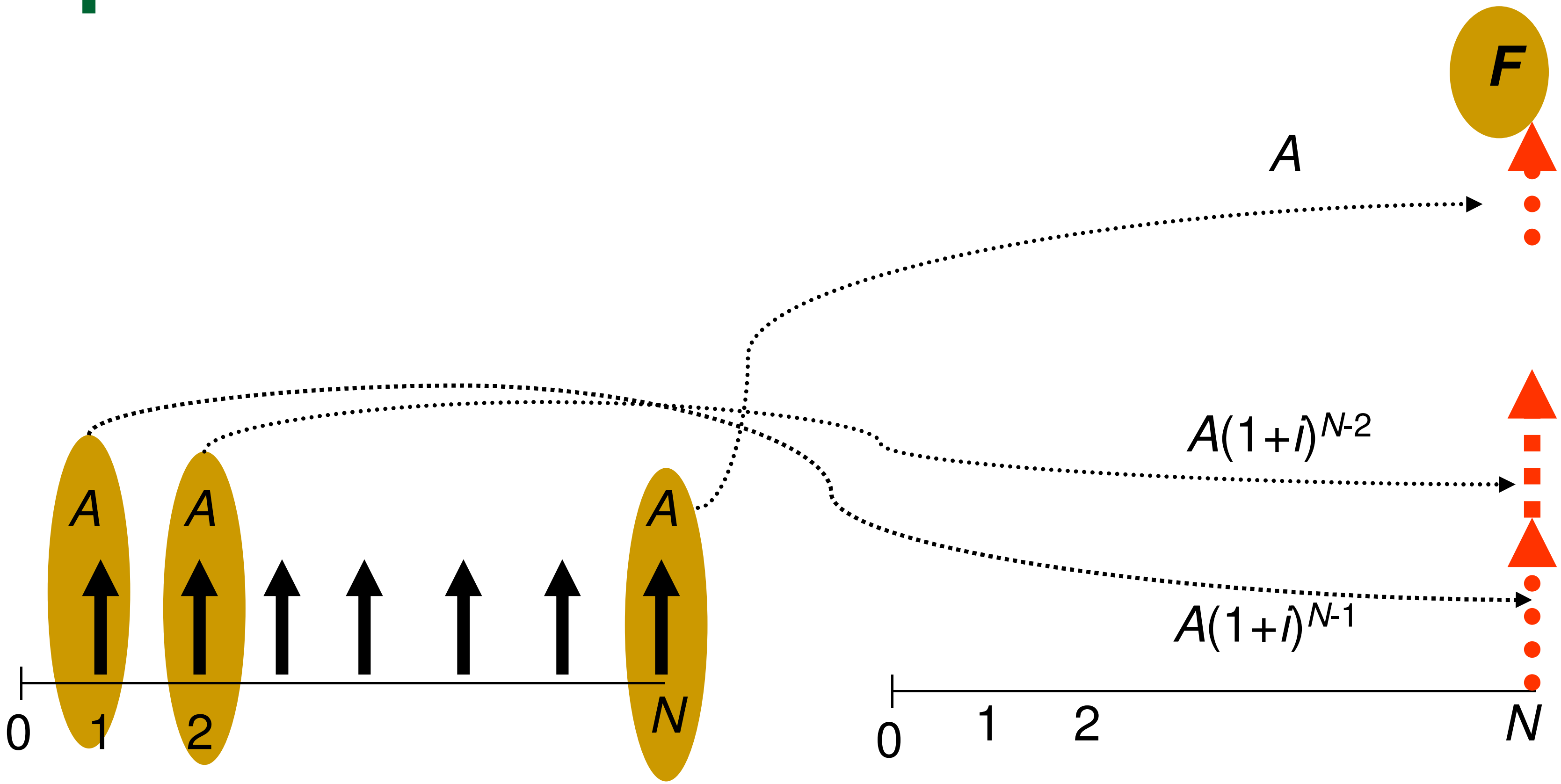
Find equivalent P or F



Equal Payment Series – Compound Amount Factor



Compound Amount Factor



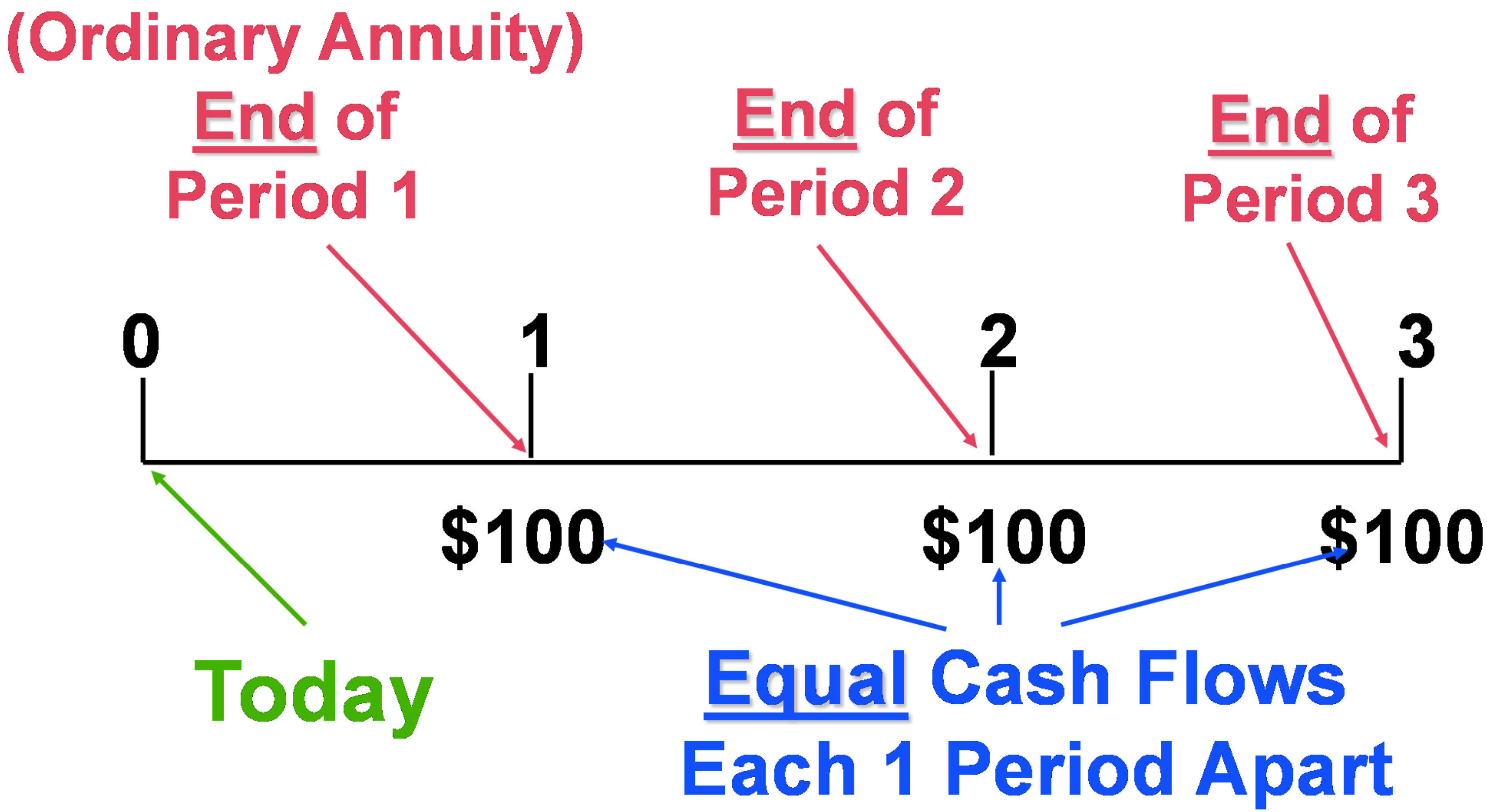
$$F = A(1+i)^{N-1} + A(1+i)^{N-2} + \cdots + A = A\left[\frac{(1+i)^N - 1}{i}\right]$$

Annuity (年金)

- An Annuity represents a series of equal payments (or receipts) occurring over a specified number of equidistant periods
- For example,
  - Student loan payments
  - Insurance premiums
  - Mortgage payments
  - Retirement savings
- A Perpetuity (永續年金) is an annuity that has no end
  - A stream of cash payments continues forever

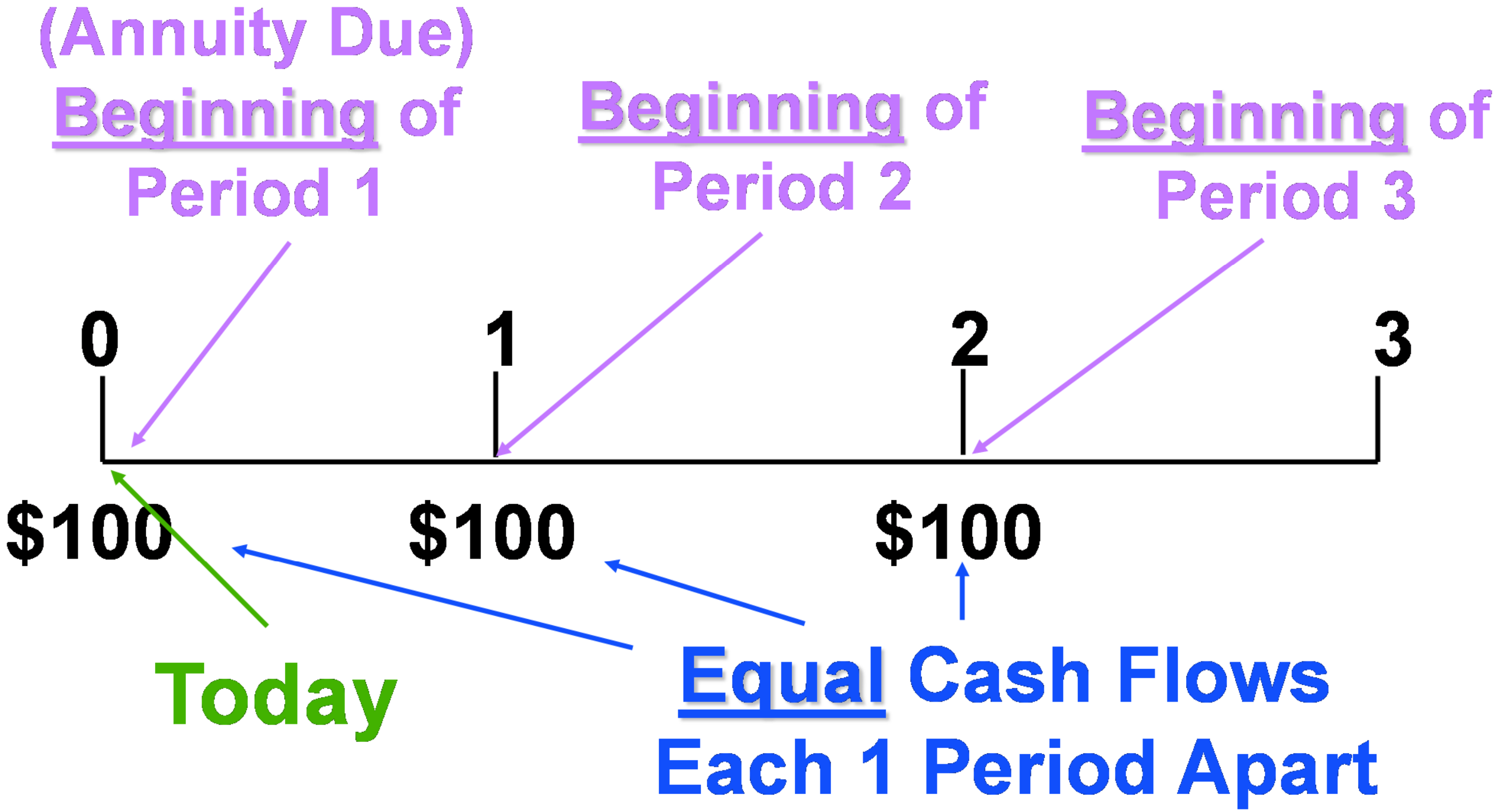
Annuity (年金)

- Ordinary Annuity: Payments or receipts occur at the end of each period

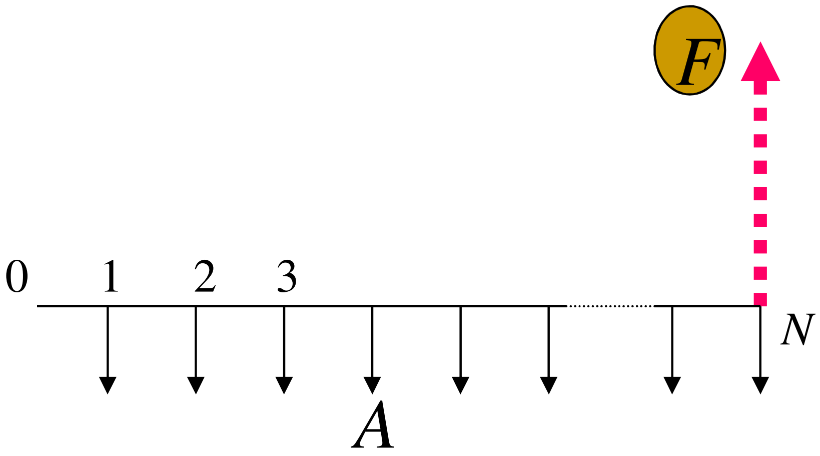


Annuity (年金)

- Annuity Due: Payments or receipts occur at the beginning of each period



Equal Payment Series Compound Amount Factor
(Future Value of an annuity) (Find *F*, Given *A*, *i*, and *N*)



$$F = A\boxed{\frac{(1+i)^N - 1}{i}}$$

$$= A(F/A, i, N)$$

Example:

- Given: A = \$5,000, N = 5 years, and i = 6%
- Find: F
- Solution: F = \$5,000(F/A, 6%, 5) = \$28,185.46

Validation

$$\$5,000(1+0.06)^4 = \$6,312.38$$

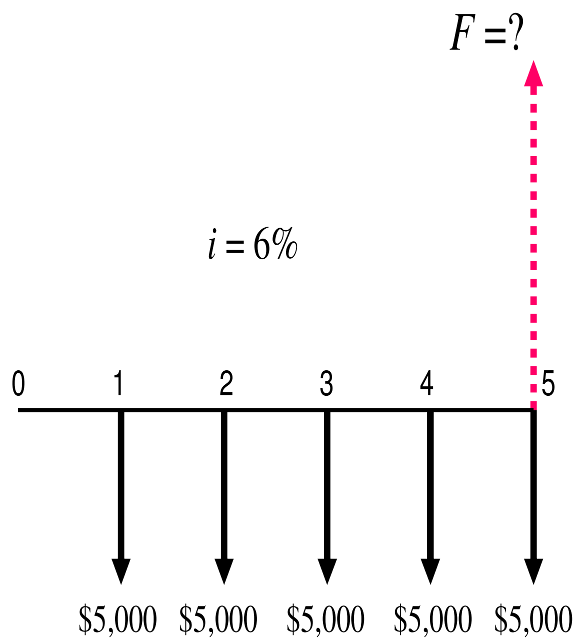
$$\$5,000(1+0.06)^3 = \$5,955.08$$

$$\$5,000(1+0.06)^2 = \$5,618.00$$

$$\$5,000(1+0.06)^1 = \$5,300.00$$

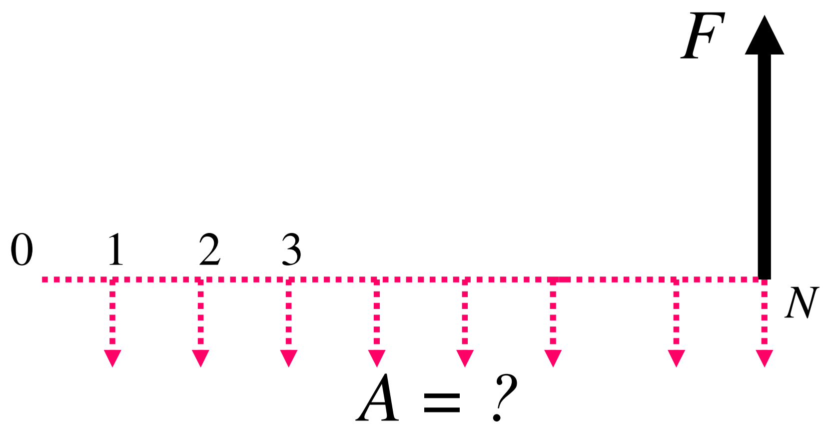
$$\$5,000(1+0.06)^0 = \$5,000.00$$

$$\boxed{\$28.185.46}$$



Finding an Annuity Value

(Find *A*, Given *F*, *i*, and *N*)



$$A = F\boxed{\frac{i}{(1+i)^N - 1}}$$

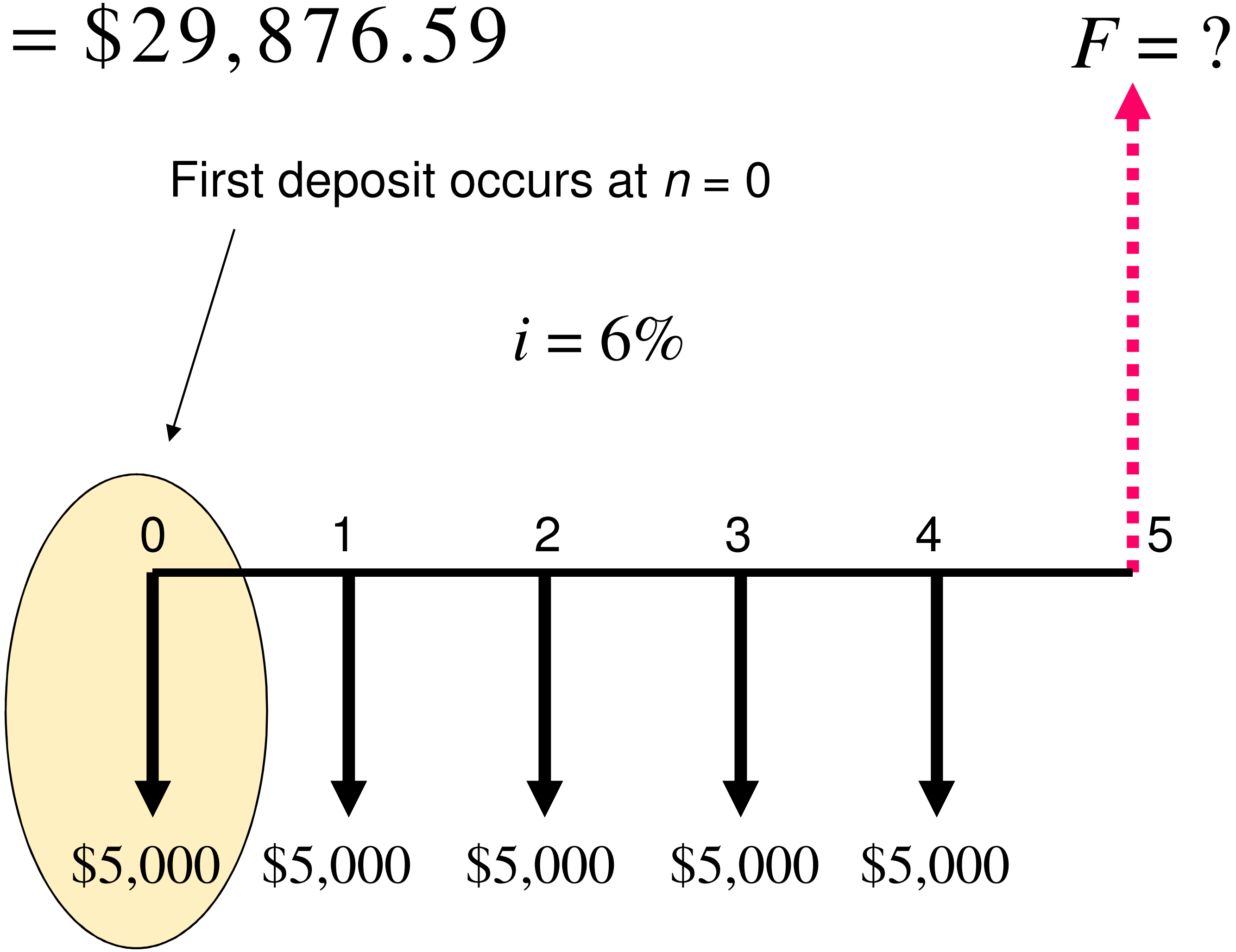
$$= F(A/F, i, N)$$

<u>Example</u>:

- Given: F = \$5,000, N = 5 years, and i = 7%
- Find: A
- Solution: A = \$5,000(A/F, 7%, 5) = \$869.50

Example: Handling Time Shifts in a Uniform Series*
(Find *F*, Given *i*, *A*, and *N*)

$$F_5 = \$5{,}000(F/A, 6\%, 5)(1.06)$$
$$= \$29{,}876.59$$



* Each payment has been shifted to one year earlier, thus each payment would be compounded for one extra year.

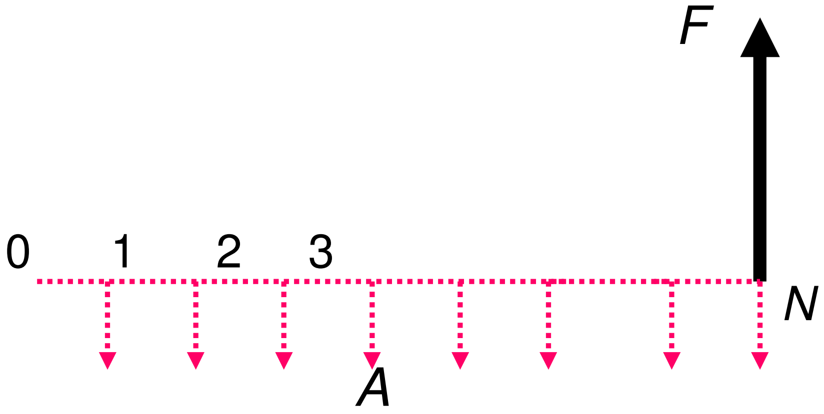
Sinking fund

(1) A fund accumulated by periodic deposits and reserved exclusively for a specific purpose, such as retirement of a debt.

(2) A fund created by making periodic deposits (usually equal) at compound interest in order to accumulate a given sum at a given future time for some specific purpose.

Sinking Fund Factor

is an interest-bearing account into which a fixed sum is deposited each interest period; The term within the colored area is called sinking-fund factor. (Find *A*, Given *F*, *i*, and *N*)



$$A = F\boxed{\frac{i}{(1+i)^N - 1}}$$

$$= F(A/F, i, N)$$

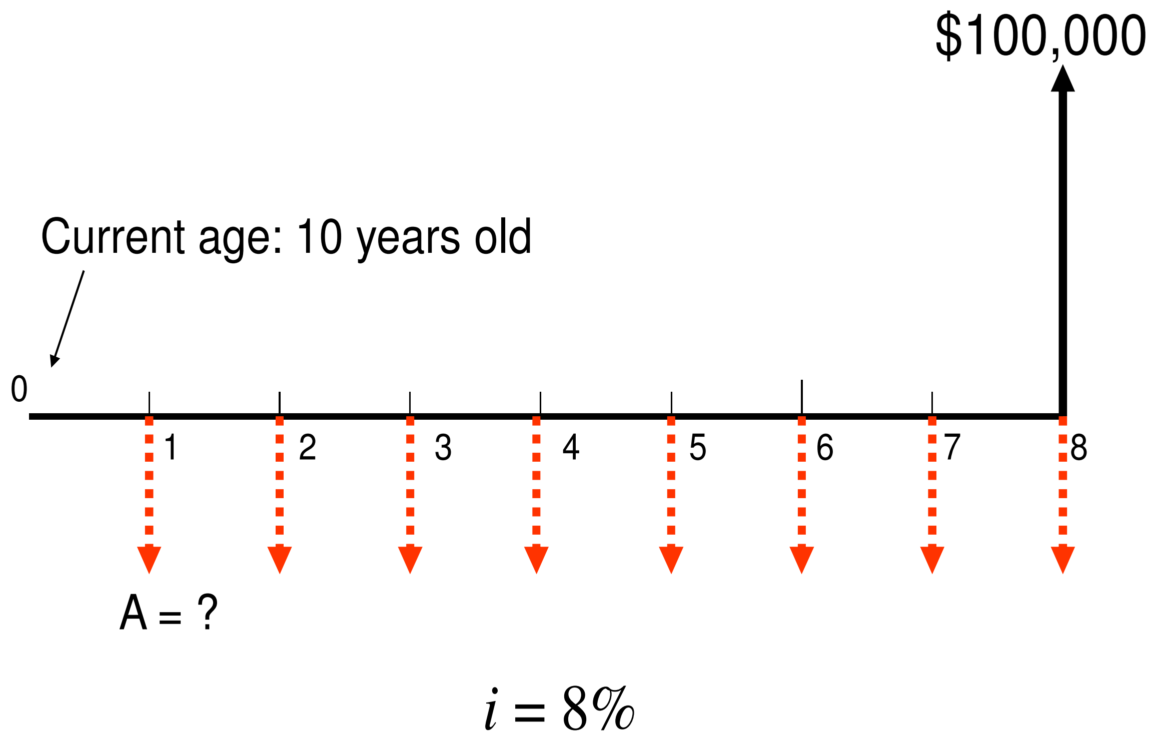
Example – College Savings Plan:

- Given: F = \$100,000, N = 8 years, and i = 7%
- Find: A
- Solution:

  A = \$100,000(A/F, 7%, 8) = \$9,746.78

OR

- **Given**:
  - ***F* = $100,00**
  - ***i* = 7%**
  - ***N* = 8 years**



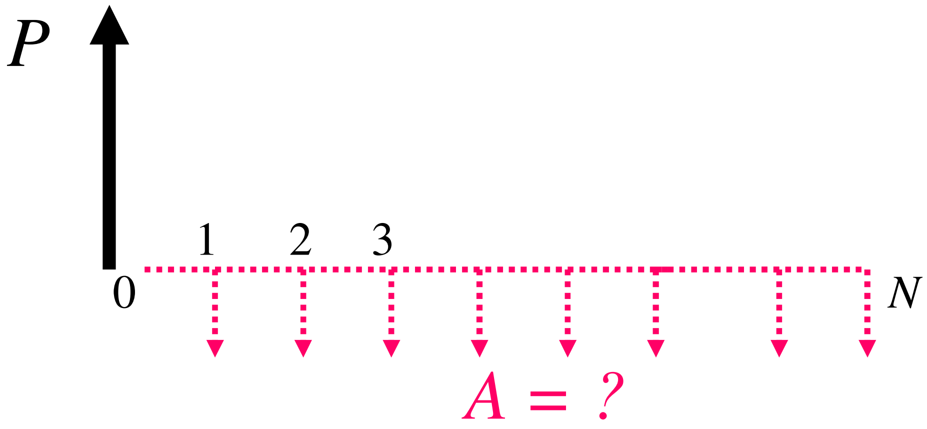
- Find: *A*
- Solution: $A = \$100{,}000(A/F, 7\%, 8) = \$9{,}746.78$

Capital Recovery Factor
(Annuity Factor)

- **Annuity:** (1) An amount of money payable to a recipient at regular intervals for a prescribed period of time out of a fund reserved for that purpose. (2) A series of equal payments occurring at equal periods of time. (3) Amount paid annually, including reimbursement of borrowed capital and payment of interest.

- **Annuity factor:** The function of interest rate and time that determines the amount of periodic annuity that may be paid out of a given fund.

Capital Recovery Factor **is the colored area which is designated (A/P, i, N). In finance, this A/P factor is referred to as the annuity factor**. (Find *A*, Given *P*, *i*, and *N*)



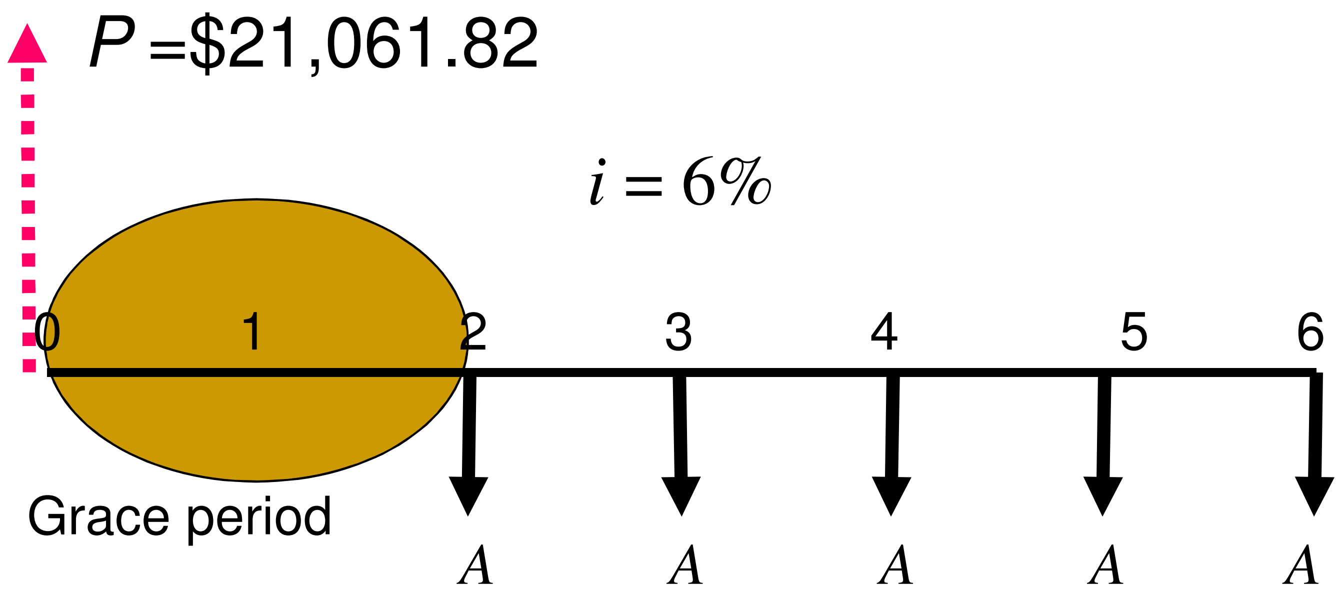
$$A = P\boxed{\frac{i(1+i)^N}{(1+i)^N - 1}}$$

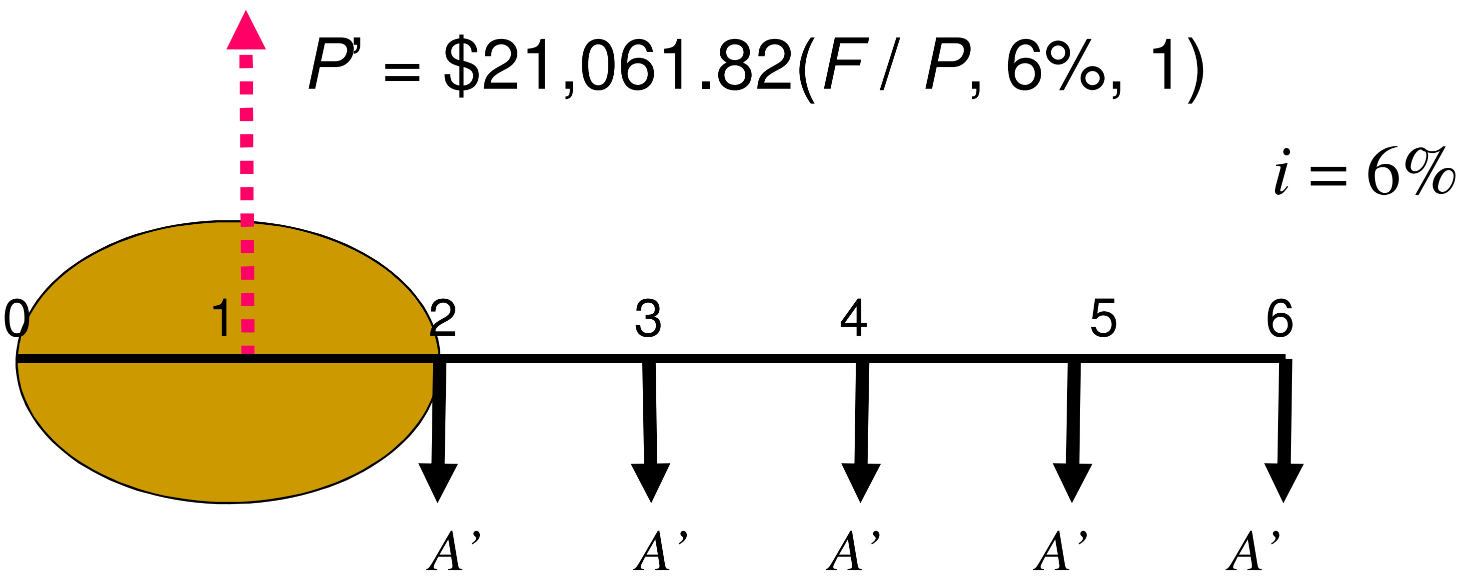
$$= P(A/P, i, N)$$

<u>Example 2.12</u>: Paying Off Education Loan

- Given: P = \$21,061.82, N = 5 years, and i = 6%
- Find: A
- Solution: A = \$21,061.82(A/P,6%,5) = \$5,000

Example: Deferred (delayed) Loan Repayment Plan





Two-Step Procedure

$$P' = \$21,061.82(F/P,6\%,1)$$

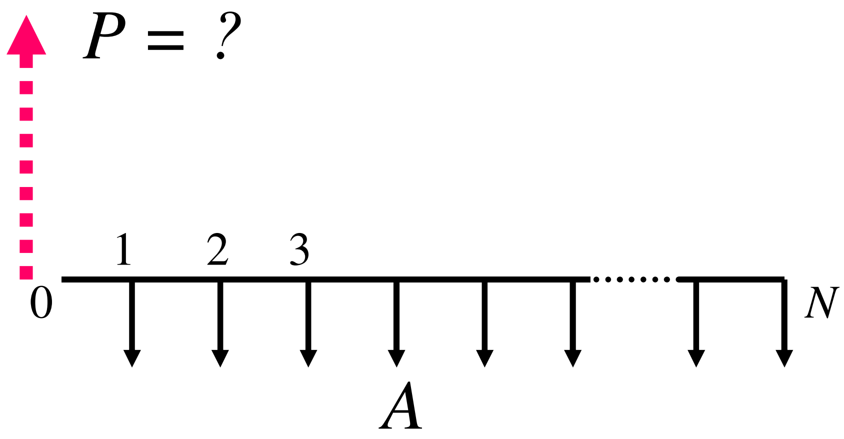
$$= \$22,325.53$$

$$A = \$22,325.53(A/P,6\%,5)$$

$$= \$5,300$$

Present Worth of Annuity Series

The colored area is referred to as the equal-payment-series present-worth factor (PWF)



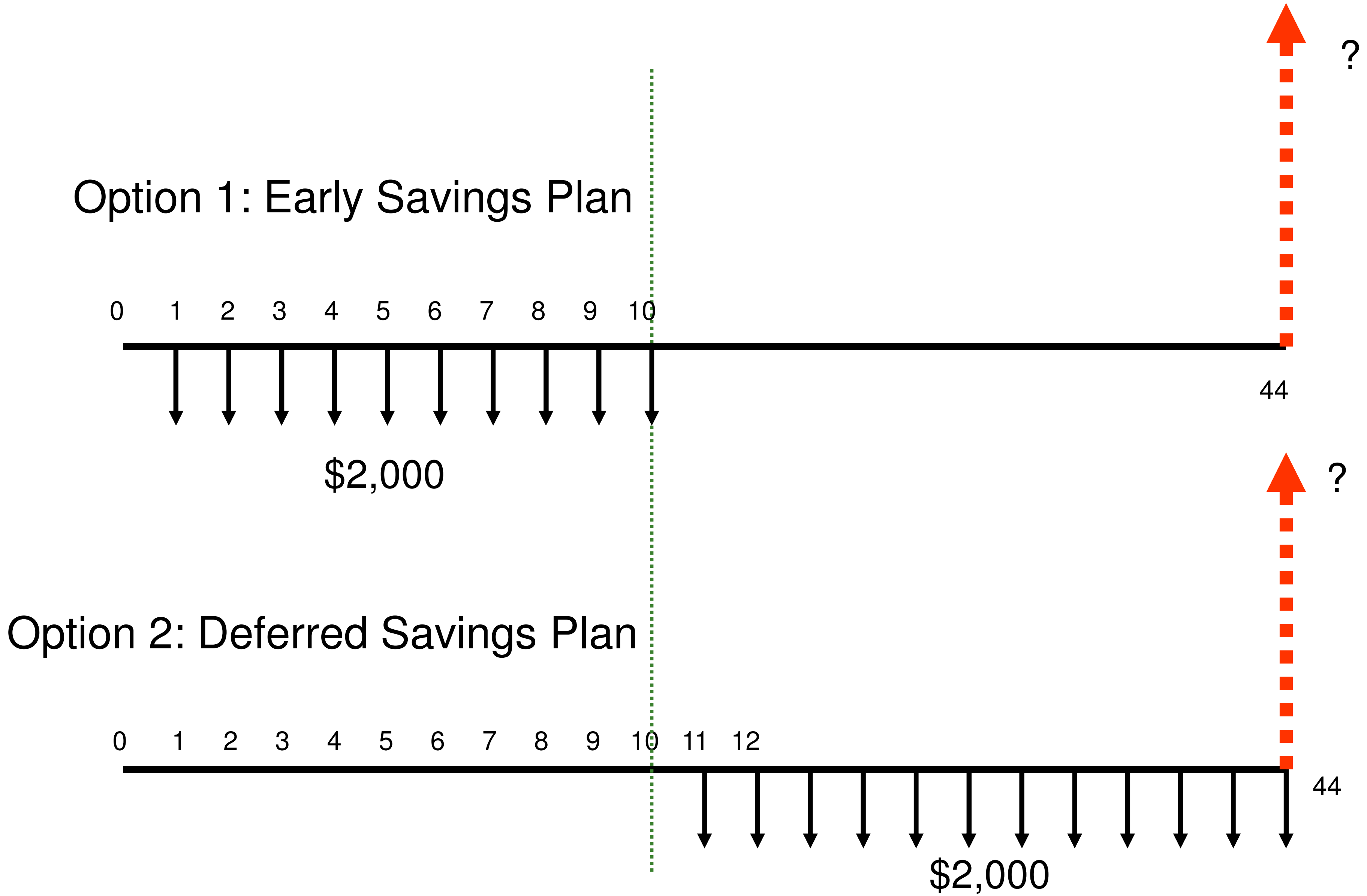
$$P = A\boxed{\frac{(1+i)^N - 1}{i(1+i)^N}}$$

$$= A(P/A, i, N)$$

Example: Lottery

- Given: A = \$7.92M, N = 25 years, and i = 8%
- Find: P
- Solution: P = \$7.92M(P/A, 8%, 25) = \$84.54M

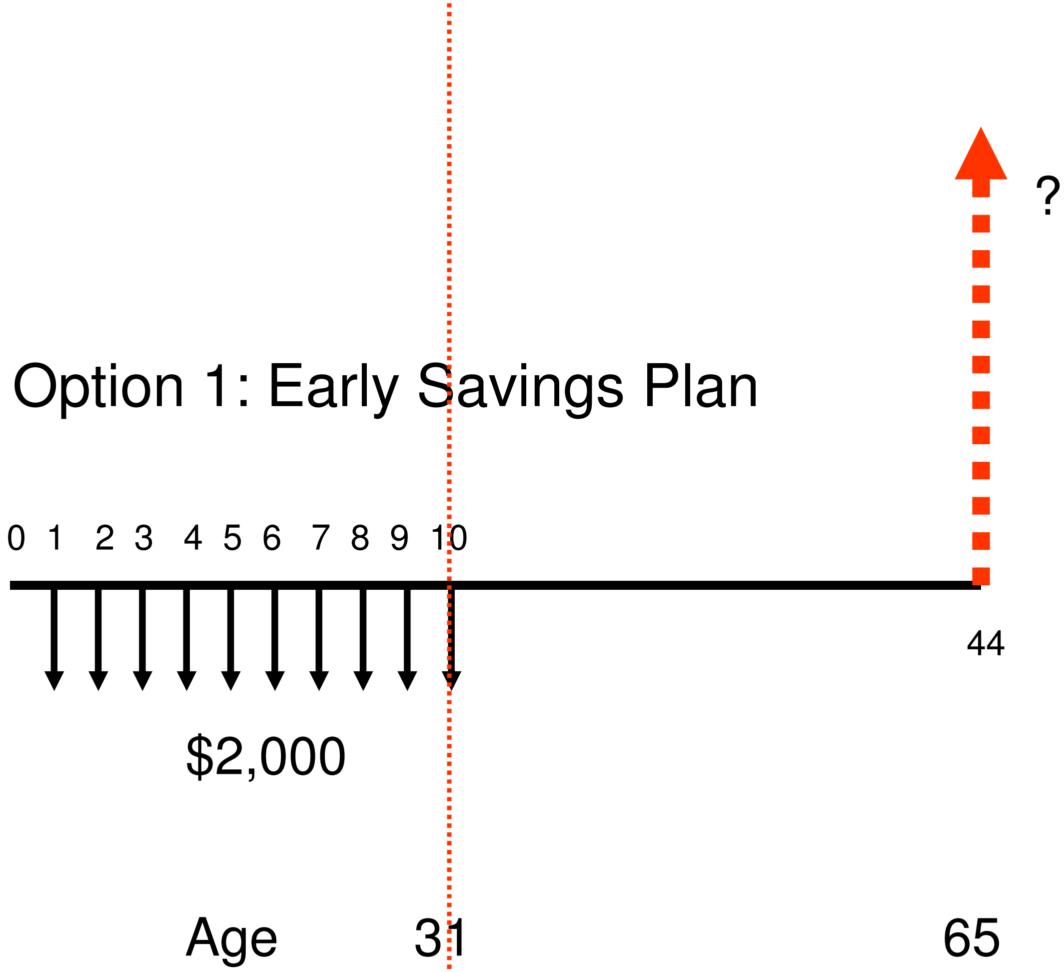
Example: Early Savings Plan – 8% interest



Option 1 – Early Savings Plan

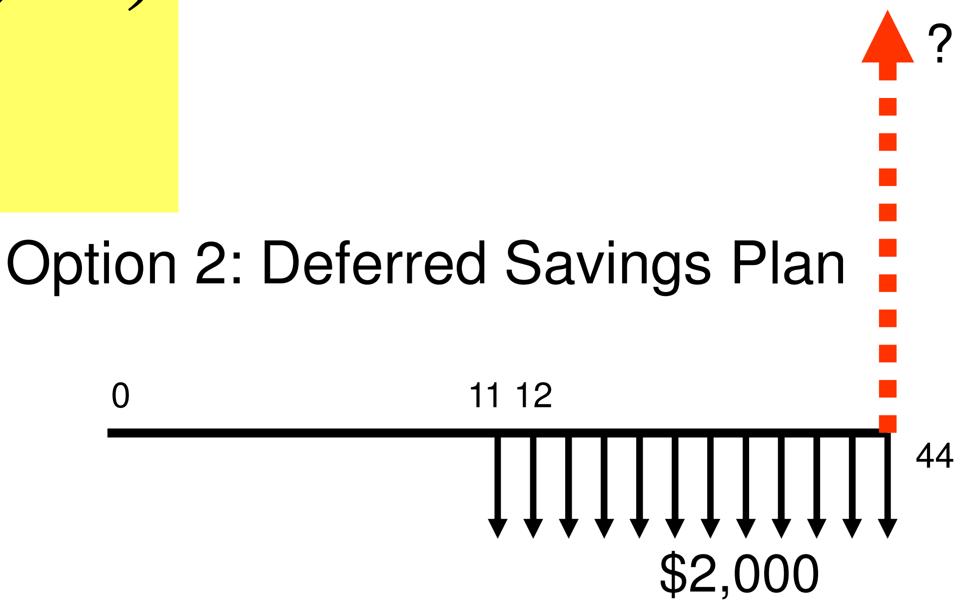
$$F_{10} = \$2,000(F/A, 8\%, 10)$$
$$= \$28,973$$

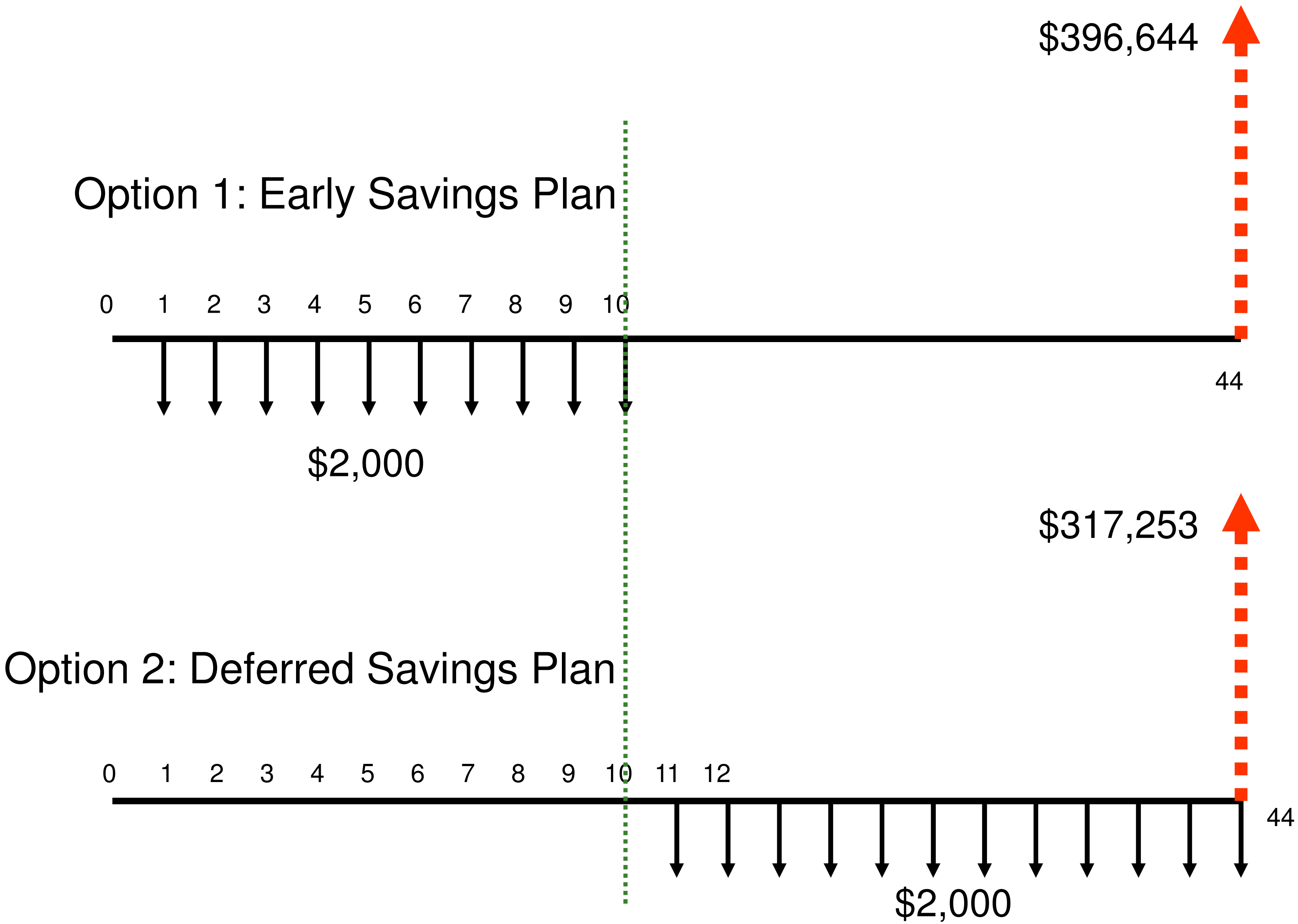
$$F_{44} = \$28,973(F/P, 8\%, 34)$$
$$= \$396,645$$



Option 2: Deferred Savings Plan

$$F_{44} = \$2{,}000(F/A, 8\%, 34)$$
$$= \$317{,}233$$



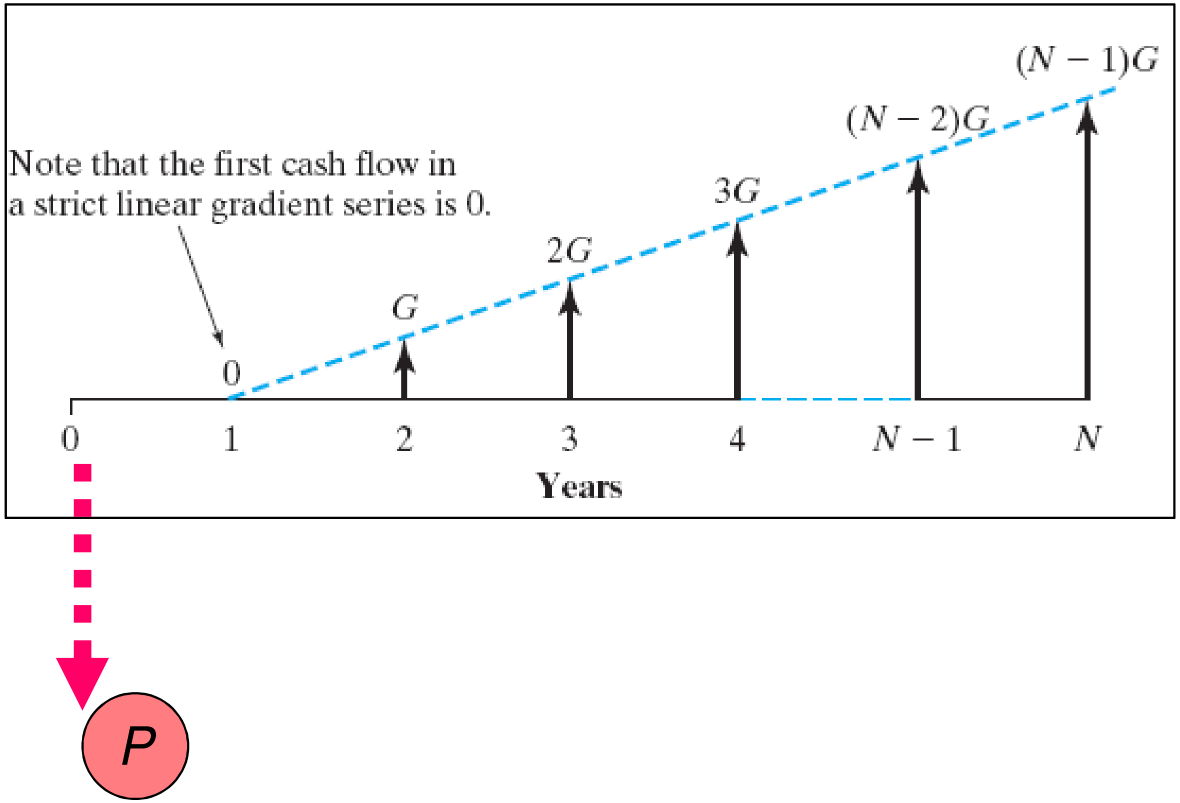


$396,644
Option 1: Early Savings Plan
0 1 2 3 4 5 6 7 8 9 10
44
$2,000
$317,253
Option 2: Deferred Savings Plan
0 1 2 3 4 5 6 7 8 9 10 11 12
44
$2,000

Linear Gradient Series

Engineers frequently meet situations involving periodic payments that increase or decrease by a constant amount (G) from period to period.

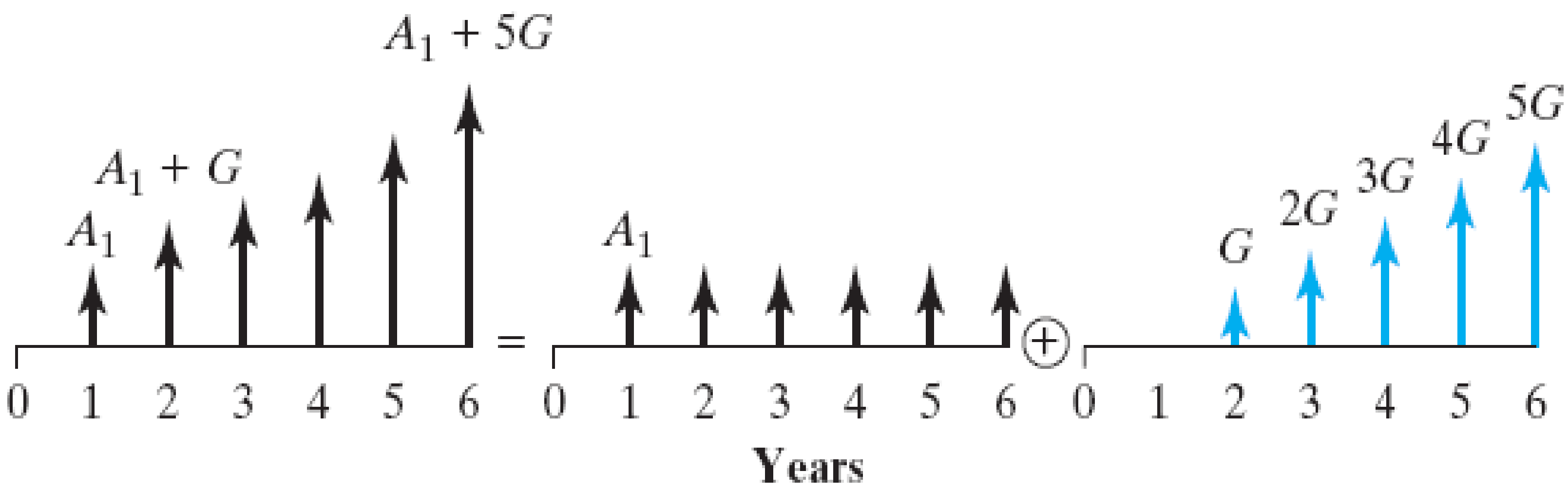
A Strict Gradient Series:



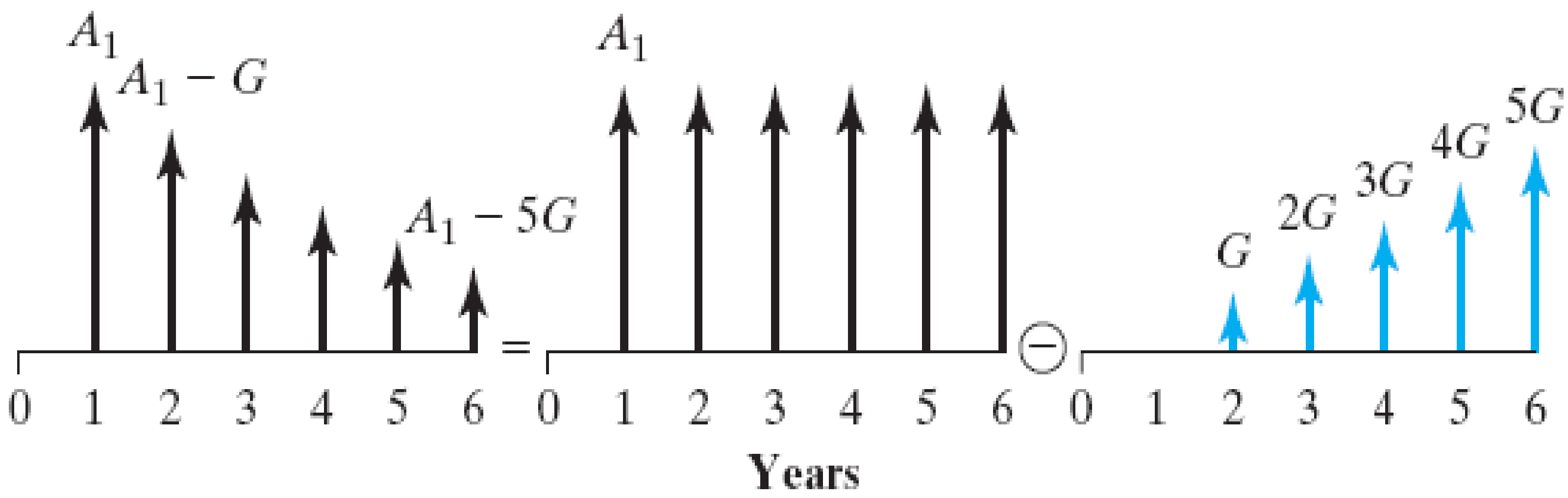
$$P = G\boxed{\frac{(1+i)^N - iN - 1}{i^2(1+i)^N}}$$

$$= G(P/G, i, N)$$

Gradient Series as a Composite Series of a Uniform Series of N Payments of A_1 and the Gradient Series of Increments of Constant Amount G

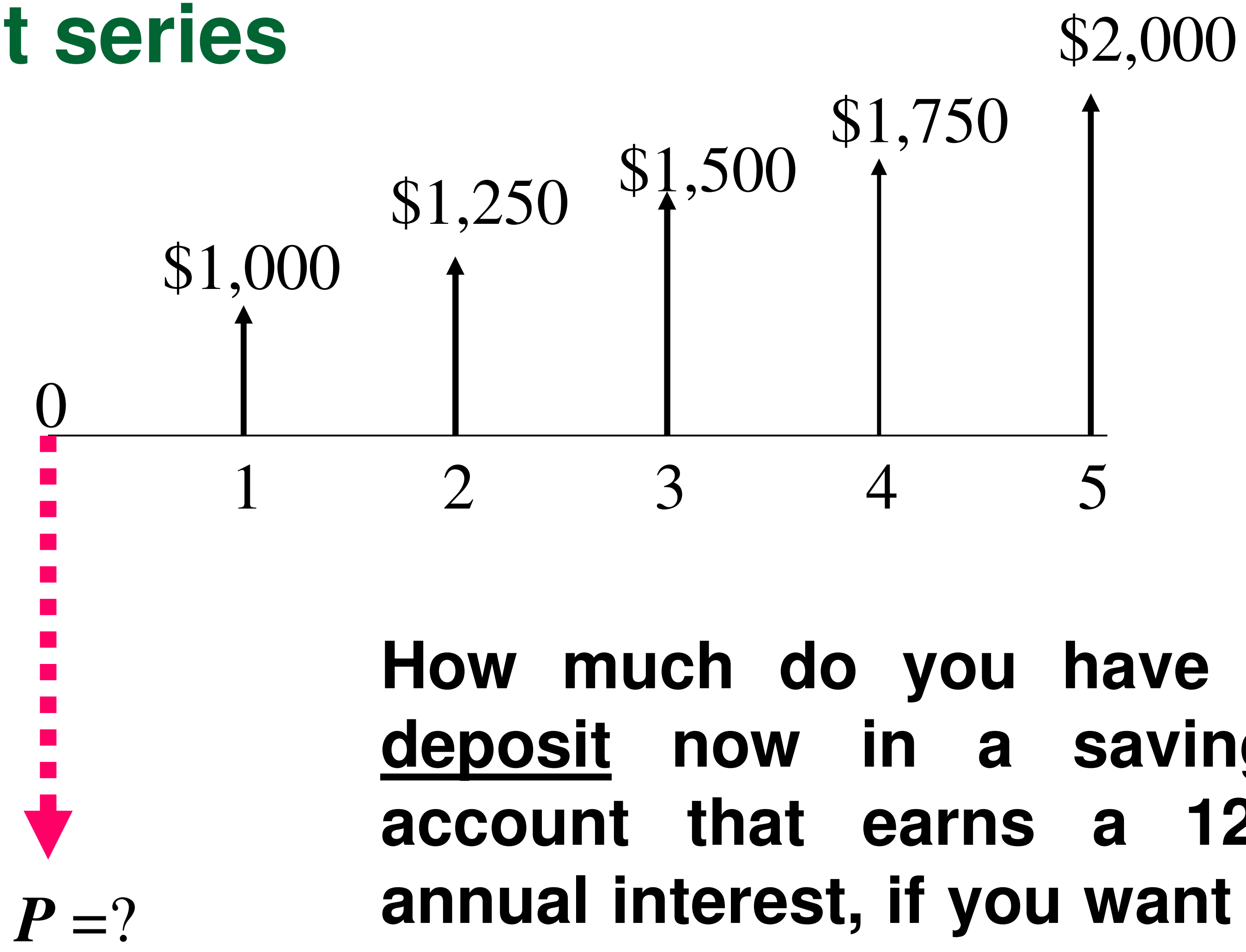


(a) Increasing gradient series



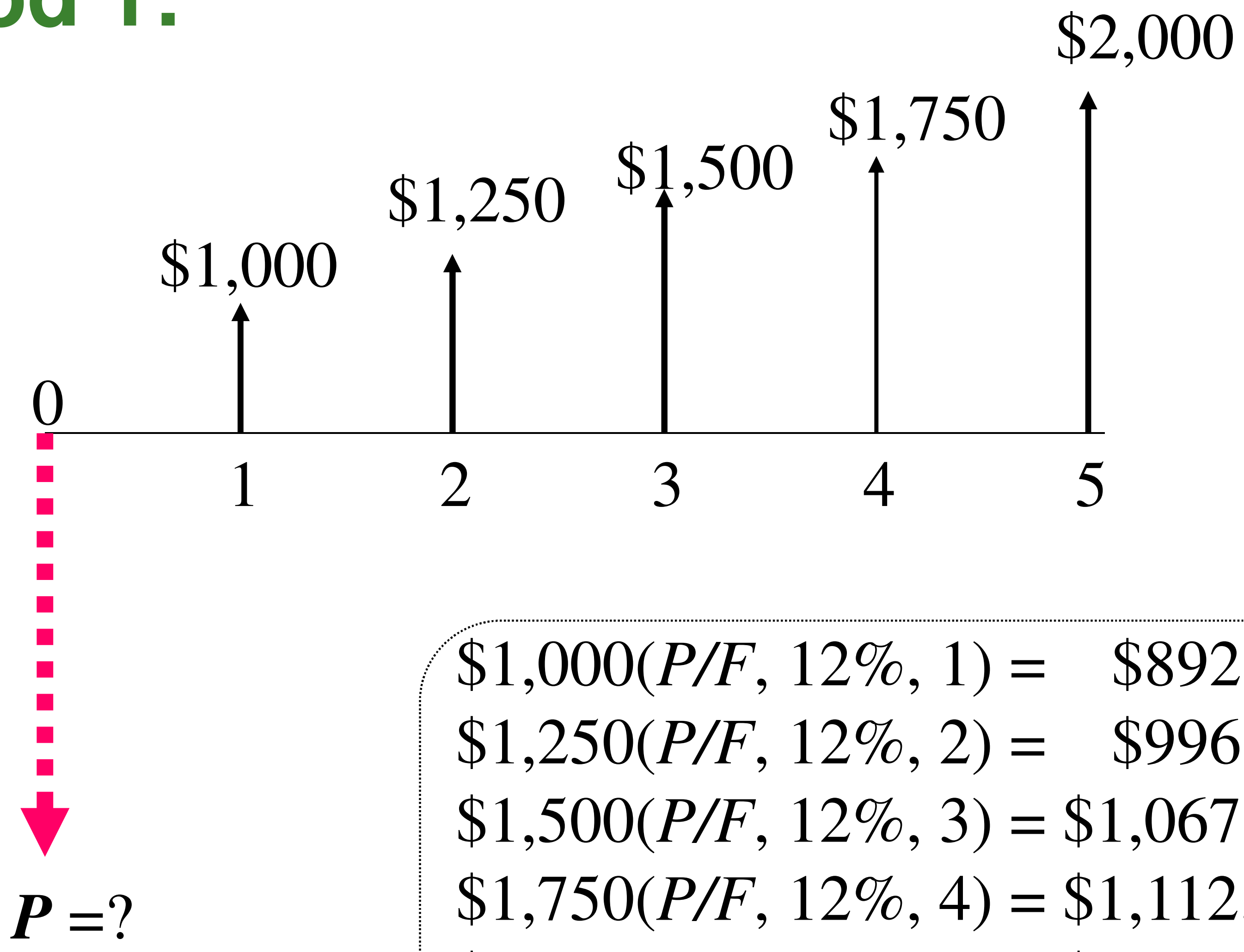
(b) Decreasing gradient series

Example – Present value calculation for a gradient series



How much do you have to <u>deposit</u> now in a savings account that earns a 12% annual interest, if you want to withdraw the annual series as shown in the figure?

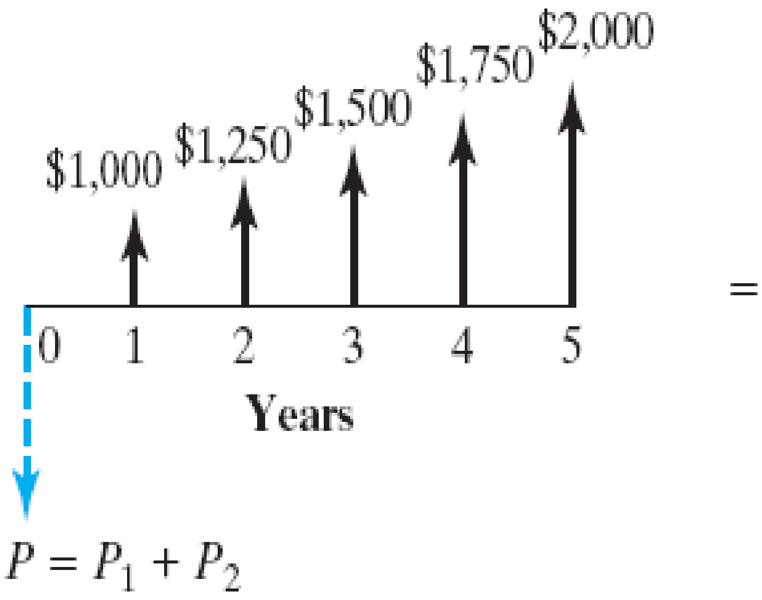
Method 1:

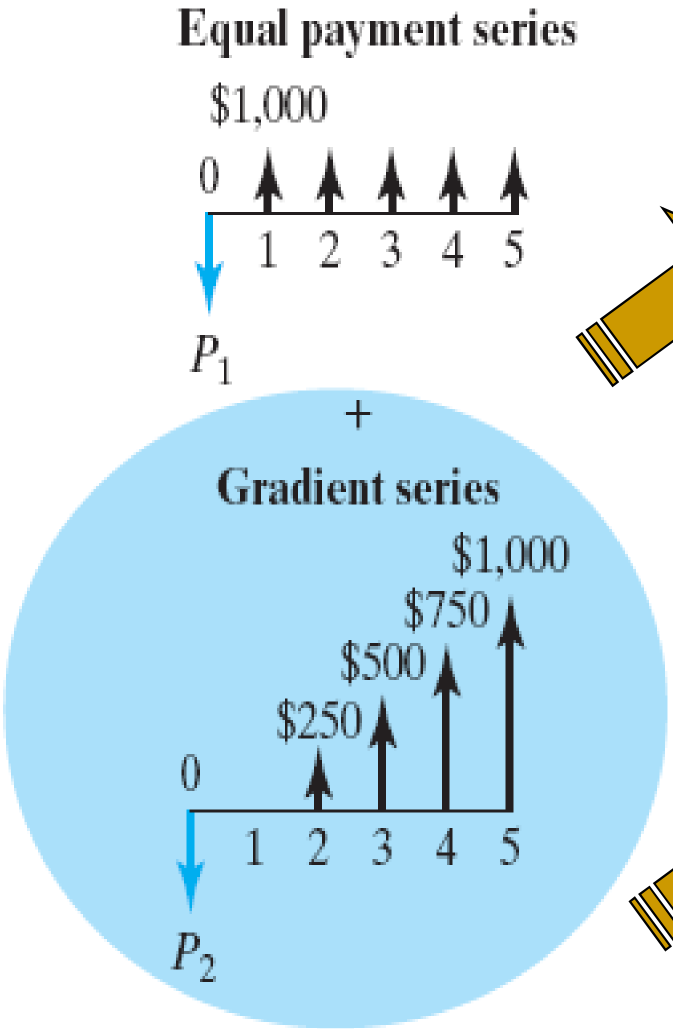


$1,000(*P/F*, 12%, 1) = $892.86
$1,250(*P/F*, 12%, 2) = $996.49
$1,500(*P/F*, 12%, 3) = $1,067.67
$1,750(*P/F*, 12%, 4) = $1,112.16
$2,000(*P/F*, 12%, 5) = $1,134.85

$5,204.03

Method 2:



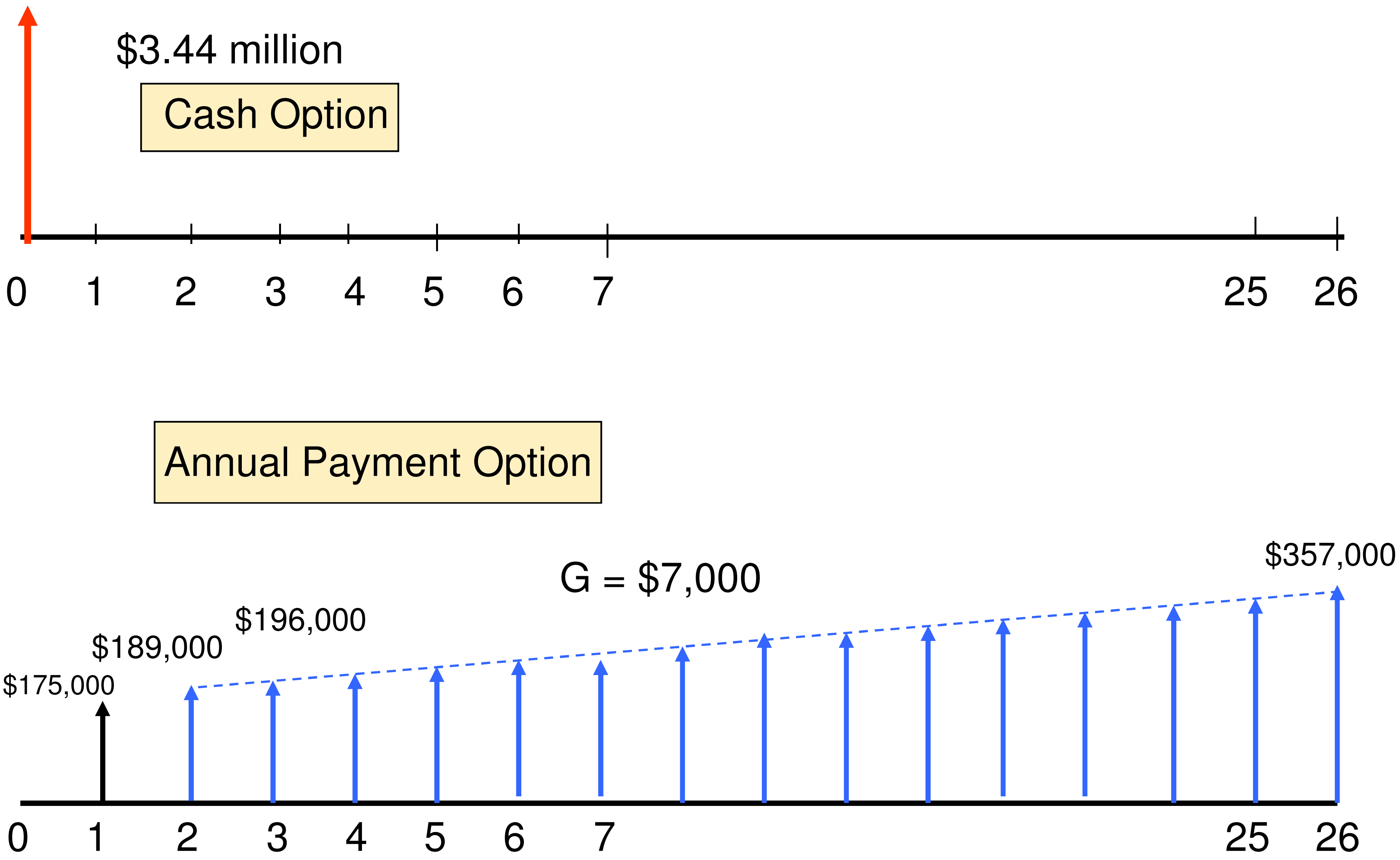


$$P_1 = \$1{,}000(P/A, 12\%, 5)$$
$$= \$3{,}604.80$$

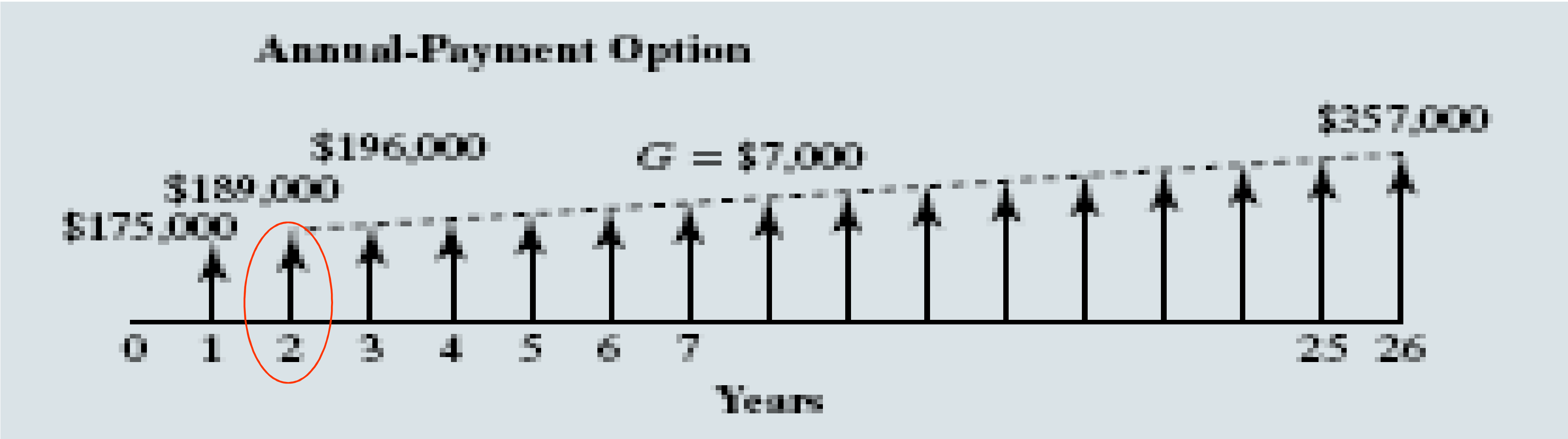
$$P_2 = \$250(P/G, 12\%, 5)$$
$$= \$1{,}599.20$$

$$P = \$3{,}604.08 + \$1{,}599.20$$
$$= \$5{,}204$$

Example: Super Lottery



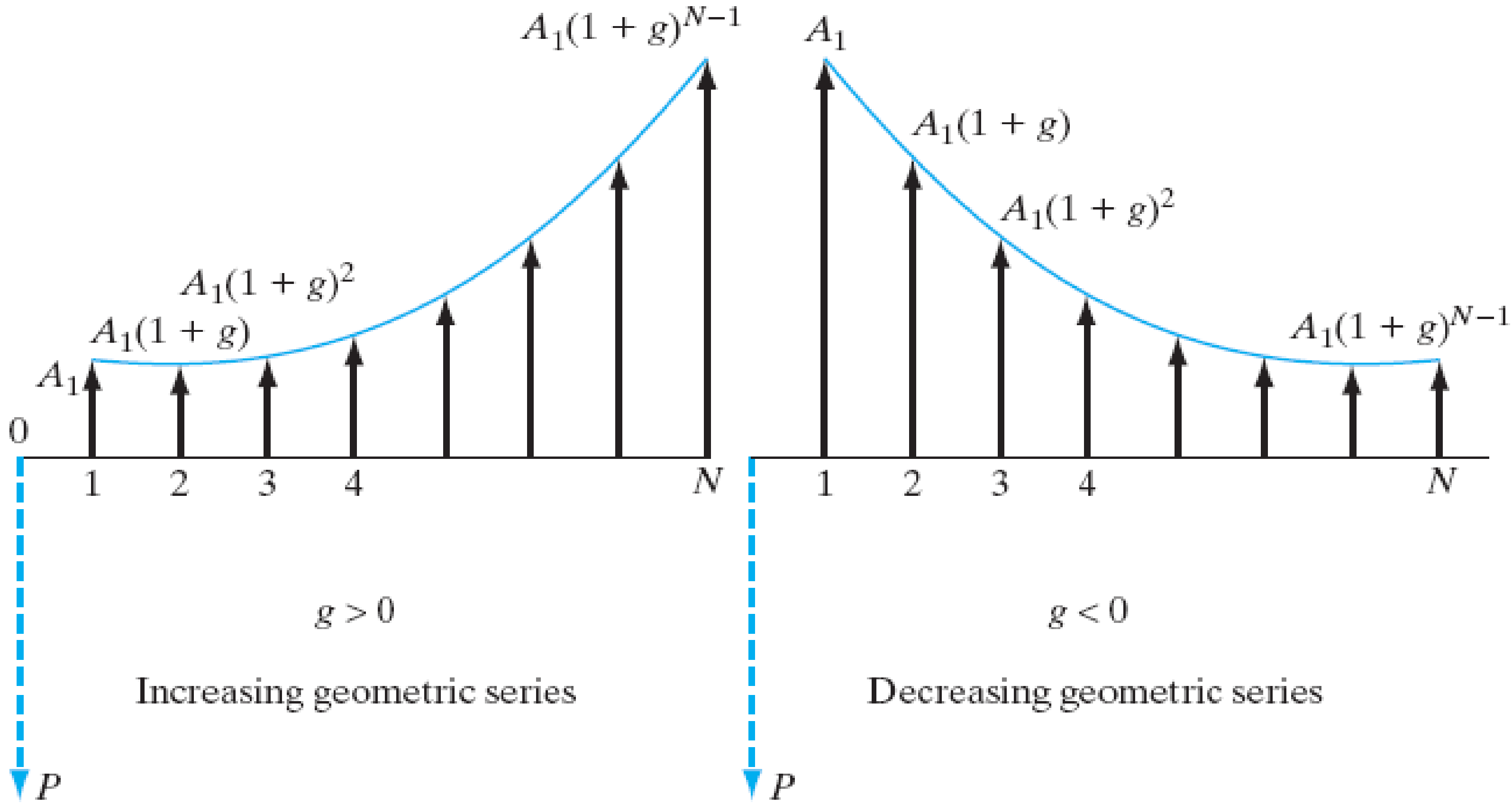
Equivalent Present Value of Annual Payment Option at 4.5%



$$P = [\$175,000 + \$189,000(P/A, 4.5\%, 25)$$
$$+\$7,000(P/G, 4.5\%, 25)](P/F, 4.5\%, 1)$$
$$= \$3,818,363$$

Geometric Gradient Series

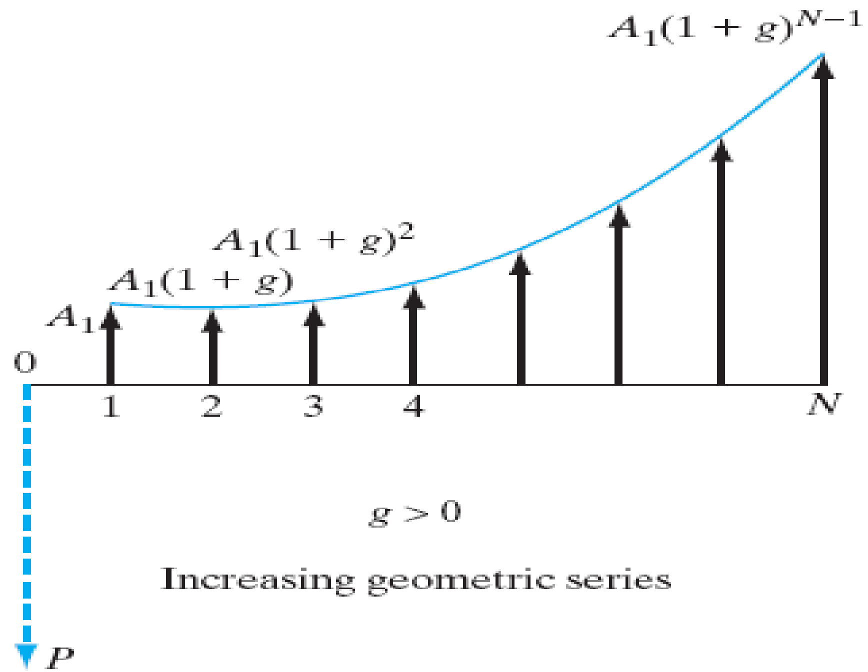
Many engineering economic problems, particularly those relating to construction costs, involve cash flows that increase over time, not by a constant amount, but rather by a constant percentage (geometric), called compound growth.



Present Worth Factor of Geometric Gradient Series

$$P = \begin{cases} A_1 \dfrac{1-(1+g)^N (1+i)^{-N}}{i-g}, & \text{if } i \neq g \\ NA_1/(1+i), & \text{if } i = g \end{cases}$$

$$= A_1\ (P/A_1, g, i, N)$$



Alternate Way of Calculating *P*

$$\text{Let } g' = \frac{i - g}{1 + g}$$

$$P = \frac{A_1}{(1+g)} (P/A, g', N)$$

Example (1): Find P, Given A_1, g, i, N
(Expected retirement pension)

- Given:
  - g = 5%
  - i = 7%
  - N = 25 years
  - A_1 = \$50,000
- Find: P

$$P = \$50,000\left[\frac{1-(1+0.05)^{25}(1+0.07)^{-25}}{0.07-0.05}\right]$$
$$= \$940,696$$

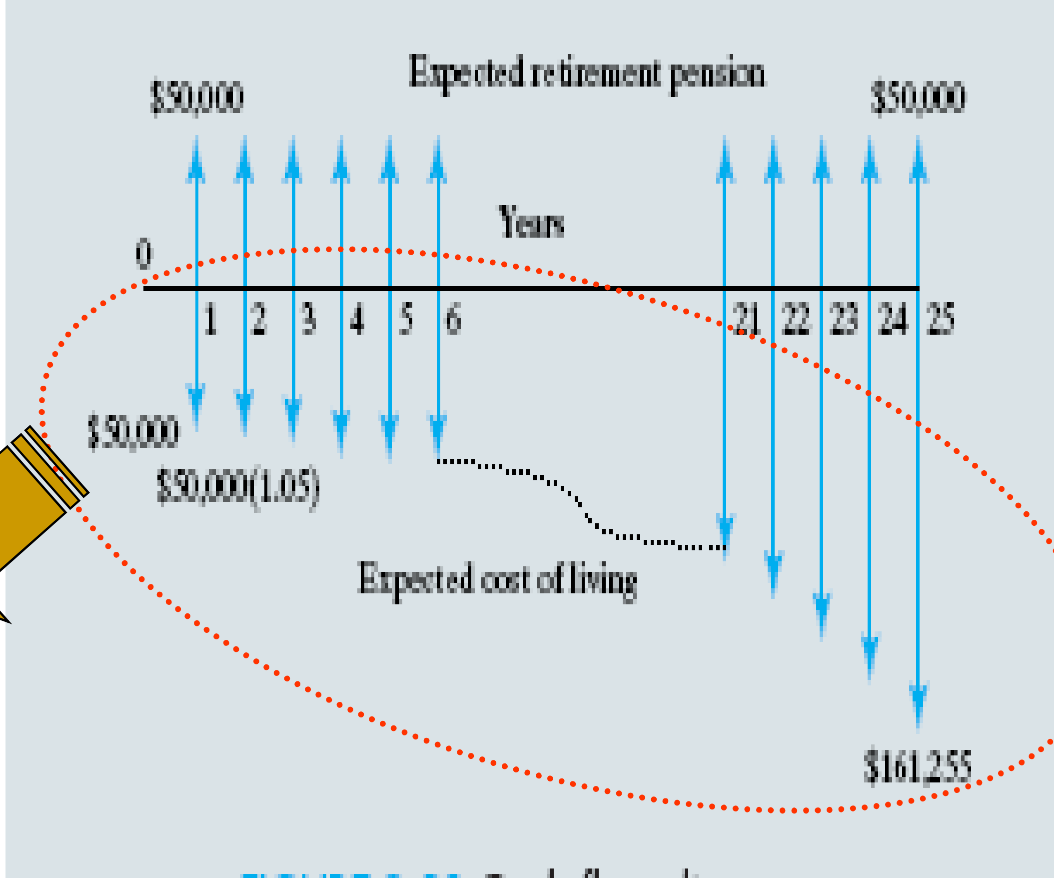


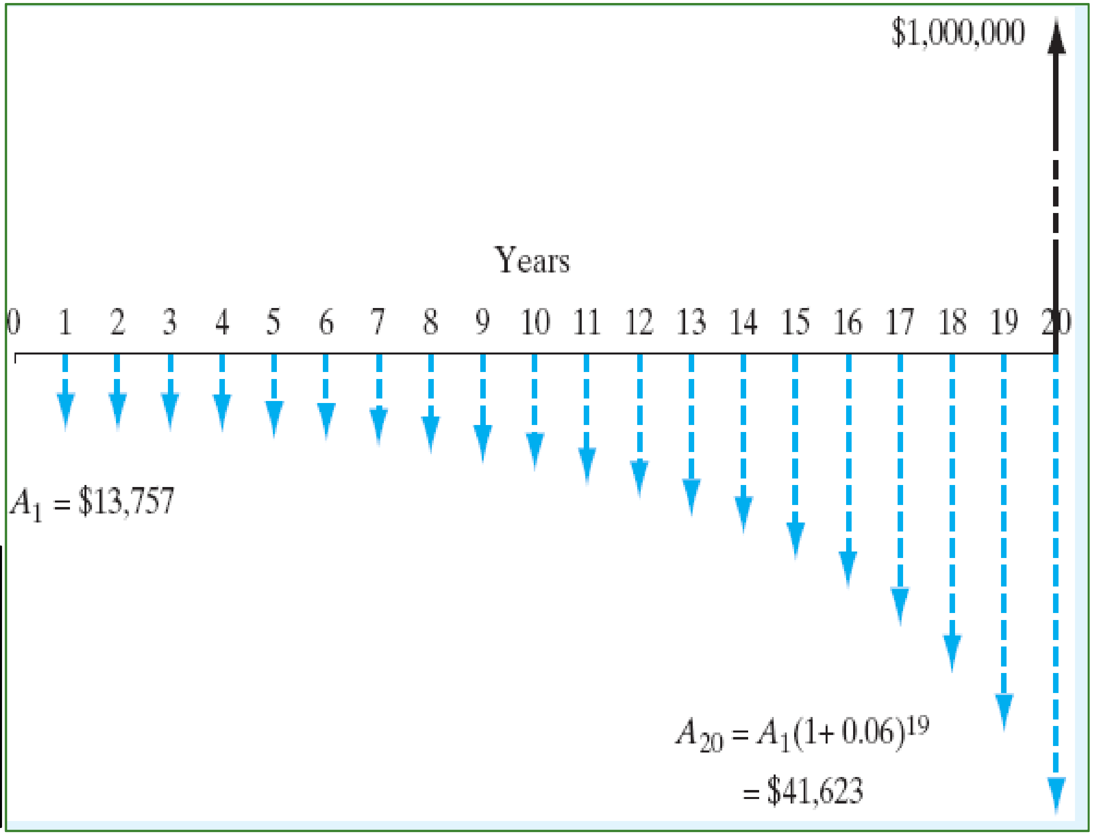
FIGURE 2.30 Cash flow diagram

Required Additional Savings

$$P = \$50,000(P/A, 7\%, 25)$$
$$= \$582,679$$
$$\Delta P = \$940,696 - \$582,679$$
$$= \$358,017$$

Example (2): Find A_1, Given F, g, i, N
(Retirement plan – saving $1 Million)

- Given:
  - F = $1,000,000
  - g = 6%
  - i = 8%
  - N = 20 years
- Find: A_1



$$
\begin{aligned}
F &= A_1(P/A_1, 6\%, 8\%, 20)(F/P, 8\%, 20) \\
&= \frac{A_1}{0.08 - 0.06}\left[1 - \left(\frac{1 + 0.06}{1 + 0.08}\right)^{20}\right](F/P, 8\%, 20) \\
&= A_1(72.6911)
\end{aligned}
$$

$$A_1 = \$1{,}000{,}000/72.6911 = \$13{,}757$$

A Typical Compound Interest Table – say 12%

- To find the compound interest factor when the interest rate is 12% and the number interest periods is 10, we could evaluate the following equation using the interest table.

	Single Payment		Equal Payment Series				Gradient Series		
N	Compound Amount Factor (F/P,i,N)	Present Worth Factor (P/F,i,N)	Compound Amount Factor (F/A,i,N)	Sinking Fund Factor (A/F,i,N)	Present Worth Factor (P/A,i,N)	Capital Recovery Factor (A/P,i,N)	Gradient Uniform Series (A/G,i,N)	Gradient Present Worth (P/G,i,N)	N
1	1.1200	0.8929	1.0000	1.0000	0.8929	1.1200	0.0000	0.0000	1
2	1.2544	0.7972	2.1200	0.4717	1.6901	0.5917	0.4717	0.7972	2
3	1.4049	0.7118	3.3744	0.2963	2.4018	0.4163	0.9246	2.2208	3
4	1.5735	0.6355	4.7793	0.2092	3.0373	0.3292	1.3589	4.1273	4
5	1.7623	0.5674	6.3528	0.1574	3.6048	0.2774	1.7746	6.3970	5
6	1.9738	0.5066	8.1152	0.1232	4.1114	0.2432	2.1720	8.9302	6
7	2.2107	0.4523	10.0890	0.0991	4.5638	0.2191	2.5515	11.6443	7
8	2.4760	0.4039	12.2997	0.0813	4.9676	0.2013	2.9131	14.4714	8
9	2.7731	0.3606	14.7757	0.0677	5.3282	0.1877	3.2574	17.3563	9
10	$\boxed{3.1058}$	0.3220	17.5487	0.0570	5.6502	0.1770	3.5847	20.2541	10

$$F = \$20{,}000\underbrace{(1 + 0.12)^{10}}_{3.1058} = \$62{,}116$$

Further Reading

- Park, C. S., 2007. *Contemporary Engineering Economics*, 4th ed., Chapter 3: Interest Rate and Economic Equivalence, Prentice Hall, Upper Saddle River, New Jersey.
  - http://esminfo.prenhall.com/sample_chapters/park/Chapter03.pdf

- Time Value of Money Using Microsoft Excel
  - www.studyfinance.com/lessons/timevalue/timevalue.xls